MA347 – HW9

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1. Let G be a finite group. Assume $\text{Aut}(G) = \{I_G\}$. Prove that G is abelian and $x^2 = e \ \forall x \in G$.

   Proof. $\forall a \in G$, then the left conjugation map $c_a : G \to G$ defined by $c_a(x) = axa^{-1}$ lies in $\text{Inn}(G) \subseteq \text{Aut}(G) = \{I_G\} \Rightarrow c_a = I_G$. Thus, $\forall a, x \in G$:

   $$\begin{aligned} I_G(x) &= c_a(x) \\ x &= axa^{-1} \\ xa &= (axa^{-1})a \\ &= ax(a^{-1}a) \\ &= axe \\ &= ax \end{aligned}$$

   Thus G is abelian.

   The inverse map $f : G \to G$ defined by $f(x) = x^{-1}$ also lies in $\text{Aut}(G) = \{I_G\}$ (it is bijective since a group is closed over inverse and the inverse element is unique). Thus $f = I_G$, so $\forall x \in G \ f(x) = x^{-1} = x = I_G(x) \Rightarrow e = xx^{-1} = x^2$. $\square$

2. Let G be a cyclic group and $f : G \to G'$ be a group homomorphism. Prove that $f(G)$ is a cyclic subgroup of G'.

   Proof. We proved in class that $f(G) \leq G'$ if f is a homomorphism, so we only need to show that $f(G)$ is cyclic.

   G is cyclic $\Rightarrow G = \langle a \rangle$ for some $a \in G$. Let $x' \in f(G)$. Then $x' = f(x)$ for some $x \in G$, where $x = a^n$, $n \in \mathbb{Z}$. We also proved in class that $\forall b \in G, n \in \mathbb{Z} \ f(b^n) = f(b)^n$ when given a group homomorphism f. Thus $x' = f(a^n) = f(a)^n$. Since every element of $f(G)$ can be written in the form $x' = f(a)^n$, $f(G)$ is a cyclic group generated by $f(a)$. $\square$