MA347 – HW6

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1. (a) Let G be a group and $a \in G$. Suppose there is $n \in \mathbb{N}$ such that $x^n = e$. Prove that there is $m \in \mathbb{N}$ such that $x^{-1} = x^m$.

   If $n = 1$, then $x^1 = x = e$. Choose $m = 1$. Then $xx^1 = ee = e = ee = x^1x \Rightarrow x^{-1} = x^1$.

   Otherwise, choose $m = n - 1 \in \mathbb{N}$. Then $xx^{n-1} = x^1x^{n-1} = x^{1+n-1} = x^n = e$, and similarly $x^{n-1}x = x^{n-1}x^1 = x^{n-1+1} = x^n = e$, so $x^{-1} = x^{n-1}$.

   (b) Assume G is a finite group. Prove that given $x \in G$, there exists $n \in \mathbb{N}$ such that $x^n = e$.

   Assume not, i.e., given $x \in G$ for some finite group G, $\nexists n \in \mathbb{N}$ such that $x^n = e$. Then the set $S = \{x^1, x^2, x^3, \dots\} \subseteq G$ is infinite if all of the elements are distinct. If they are not all distinct, then $\exists$ distinct $i, j \in \mathbb{N}$ (w.l.o.g. assume $i < j$) such that $a^i = a^j \Rightarrow a^i = a^i a^{j-i} \Rightarrow a^{j-i} = e$. Then there exists $n = j - i \in \mathbb{N}$ such that $a^n = e$, but this contradicts the hypothesis.

   Thus all of the elements of S must be distinct, so it must be an infinite set, so G must be infinite-order $\Rightarrow$ contradiction of hypothesis. Thus there must $\exists n \in \mathbb{N}$ such that $x^n = e$.

2. Let G be a group such that $x^2 = e$ for all $x \in G$. Show that G is abelian.

   For all $x \in G$. By hypothesis, $xx = e = xx \Rightarrow x = x^{-1}$. Let $a, b \in G$. Then $ab = a^{-1}b^{-1} = (ba)^{-1}$ (this result was proved during lecture). Since $ba \in G$, $(ba)^{-1} = ba \Rightarrow ab = ba$.