MA347 – HW3

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(In these solutions, we take for granted that the rules of matrix multiplication (e.g., associativity), matrix inversion (e.g., if $T, U \in M_{n\times n}(\mathbb{R})$ are invertible, then $(TU)^{-1} = U^{-1}T^{-1}$), and integer addition (e.g., integers are closed over additive inverse and addition/subtraction) are known.)

1. Define $A \underset{s}{\sim} B \Leftrightarrow A = TBT^{-1}$ for some invertible $T \in M_{n\times n}(\mathbb{R})$, and $A, B \in M_{n\times n}(\mathbb{R})$. Prove that $\underset{s}{\sim}$ is an equivalence relation on $M_{n\times n}(\mathbb{R})$.

   Proof. **Reflexivity** Let $A \in M_{n\times n}(\mathbb{R})$. It is clear that $I_n \in M_{n\times n}(\mathbb{R})$, and $I_n^{-1} = I_n$. $A = IA = (IA)I = (IA)I^{-1} = IAI^{-1} \Rightarrow A \sim A$.

   Symmetry Let $A, B \in M_{n\times n}(\mathbb{R})$, and let $T \in M_{n\times n}(\mathbb{R})$ invertible. Then $A = TBT^{-1} \Rightarrow T^{-1}AT = T^{-1}TBT^{-1}T \Rightarrow IBI = B = T^{-1}AT$. Since T is invertible, then $U = T^{-1}$ is also invertible matrix in $M_{n\times n}(\mathbb{R})$, where $(T^{-1})^{-1} = T$. Thus $B = UBU^{-1}$, where $U \in M_{n\times n}(\mathbb{R})$ invertible; thus $B \sim A$.

   Transitivity Let $A, B, C \in M_{n\times n}(\mathbb{R})$, and let $T, U \in M_{n\times n}(\mathbb{R})$ invertible, such that $A = TBT^{-1}, B = UCU^{-1}$ (i.e., $A \sim B, B \sim C$). Then $A = T(UCU^{-1})T^{-1} = (TU)C(U^{-1}T^{-1})$. If we let $V = TU$, then we know that TU is in $M_{n\times n}(\mathbb{R})$ invertible and $(TU)^{-1} = U^{-1}T^{-1}$; thus $A = VCV^{-1} \Rightarrow A \sim C$.

   □

2. Define $x \sim y \Leftrightarrow x - y = n$ for some $n \in \mathbb{Z}$ and that $x, y \in \mathbb{R}$. Prove that $\sim$ is an equivalence relation on $\mathbb{R}$.

   Proof. **Reflexivity** Let $x \in \mathbb{R}$. Then $x - x = 0 \in \mathbb{Z} \Rightarrow x \sim x$.

   Symmetry Let $x \sim y$, where $x, y \in \mathbb{R}$. Then $x - y \in \mathbb{Z} \Rightarrow y - x = -(x - y) \in \mathbb{Z} \Rightarrow y \sim x$.

   Transitivity Let $x, y, z \in \mathbb{R}$, and $x \sim y$, $y \sim z$. Then $x - y = n_1 \in \mathbb{Z}$, $y - z = n_2 \in \mathbb{Z} \Rightarrow y = z + n_2 \Rightarrow x - (z + n_2) = n_1 \Rightarrow x - z = n_1 - n_2 \in \mathbb{Z} \Rightarrow x \sim z$.

   □