MA347 – HW24

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1. Let R be a ring and $I \subseteq J \subseteq R$ where I and J are two-sided ideals. Prove that there is a (unique) ring homo. $\varphi : {}^R/_I \to {}^R/_J$ such that $\varphi(a+I) = a+J$.

   Proof. First, we should show that φ is indeed a ring homo.

   Let $a+I, b+I \in {}^R/_I$, where $a, b \in R$. The product and sum of these factor rings are well defined:

$$\begin{aligned}
(a+I)(b+I) &= a(b+I) + I(b+I) && \text{(rings are distributive)} \\
&= ab + aI + bI + II && \text{(rings are distributive)} \\
&= ab + I + I + II && \text{(I is two-sided ideal)} \\
&= ab + I && (II = I = I + I) \\
(a+I) + (b+I) &= a + b + I + I && \text{(additive group is abelian)} \\
&= (a+b) + I
\end{aligned}$$

   The fact that φ is a ring homomorphism follows naturally:

$$\begin{aligned}
\varphi((a+I)(b+I)) &= f(ab+I) \\
&= ab + J \\
&= (a+J)(b+J) \\
&= \varphi(a+I)\varphi(b+I) \\
\varphi((a+I)+(b+I)) &= \varphi((a+b)+I) \\
&= (a+b) + J \\
&= (a+J) + (b+J) \\
&= \varphi(a+I) + \varphi(b+I) \\
\varphi(e) &= \varphi(0+I) \\
&= 0 + J = e'
\end{aligned}$$

   $\therefore \varphi$ is a ring homo.

   We should also show that φ is well-defined. Let $a+I = b+I$ for $a, b \in R$. Then $a - b \in I \subseteq J \Rightarrow a+J = b+J$. Thus φ is well-defined.

   This map is unique because if $\psi : {}^R/_I \to {}^R/_J$ is a ring homomorphism that maps $a+I \mapsto a+J$ for all $a+I \in {}^R/_I$, then $\psi = \varphi$. □

2. Let R be an integral domain and $f : R \to R$ is a ring automorphism. Prove that there is a unique automorphism $f^* : K \to K$ of fields such that $f^*(r) = f(r)\ \forall r \in R$ where K is the quotient field of R.

Proof. This proof is identical to the textbook example on pages 103-4 of Lang (with some additional commentary), except that f is an automorphism rather than an embedding.

To show uniqueness, let $a \neq 0 \in R$. If f^* is a homo., then we must have:

$$1 = f^*(1) = f^*\left(\frac{1}{a}\frac{a}{1}\right) = f^*\left(\frac{1}{a}\right) f^*(a)$$
$$\Rightarrow f^*\left(\frac{1}{a}\right) = [f^*(a)]^{-1} = \frac{1}{f^*(a)}$$

Now, consider the fact that $f^*(r) = f(r)\ \forall r \in R$, i.e., that f^* extends f to its fraction field. Thus $\forall a, b \in R,\ b \neq 0$ we must have:

$$f^*\left(\frac{a}{b}\right) = f^*\left(\frac{a}{1}\frac{1}{b}\right) = f^*(a)\frac{1}{f^*(b)} = \frac{f^*(a)}{f^*(b)} = \frac{f(a)}{f(b)}$$

Thus the map f^* is uniquely determined by the effect of the map f on R. Also, it is clear that f^* is an extension of f: $\forall a = a/1 \in R \subseteq K$, then $f^*(a/1) = f(a)/f(1) = f(a)$.

To show that f^* is well-defined, let $x = a/b, y = c/d \in K$ for $a, b, c, d \in R$, and $x = y$. Then

$$f^*(x) = \frac{f(a)}{f(b)}, \qquad f^*(y) = \frac{f(c)}{f(d)}$$
$$x = y \Rightarrow ad = bc$$
$$f(a)f(d) = f(ad) = f(bc) = f(b)f(c) \Rightarrow f^*(x) = f^*(y)$$

Lastly, we need to show that f^* is indeed an automorphism, which was assumed up till now. Let $a, c \in R,\ b, d \in R^*$:

$$f^*\left(\frac{a}{b}\frac{c}{d}\right) = f^*\left(\frac{ac}{bd}\right) = \frac{f(ac)}{f(bd)} = \frac{f(a)f(c)}{f(b)f(d)}$$
$$= \frac{f(a)}{f(b)}\frac{f(c)}{f(d)} f^*\left(\frac{a}{b}\right) = f^*\left(\frac{c}{d}\right)$$
$$f^*\left(\frac{a}{b} + \frac{c}{d}\right) = f^*\left(\frac{ad + bc}{bd}\right) = \frac{f(ad + bc)}{f(bd)}$$
$$= \frac{f(a)f(d) + f(b)f(c)}{f(b) + f(d)}$$
$$= \frac{f(a)}{f(b)} + \frac{f(c)}{f(d)} = f^*\left(\frac{a}{b}\right) + f^*\left(\frac{c}{d}\right)$$
$$f^*\left(\frac{1}{1}\right) = \frac{f(1)}{f(1)} = \frac{1}{1} = 1$$

$\therefore f^*$ is a homo.

To show that f is a ring auto., define a map $g^* : K \to K$ by $a/b \mapsto f^{-1}(a)/f^{-1}(b)$. Then, for $\forall a, c \in R$ and $b, d \in R^*$:

$$g^*\left(f^*\left(\frac{a}{b}\right)\right) = g^*\left(\frac{f(a)}{f(b)}\right) = \frac{f^{-1}(f(a))}{f^{-1}(f(b))} = \frac{a}{b}$$
$$f^*\left(g^*\left(\frac{a}{b}\right)\right) = f^*\left(\frac{f^{-1}(a)}{f^{-1}(b)}\right) = \frac{f(f^{-1}(a))}{f(f^{-1}(b))} = \frac{a}{b}$$

Thus $g^* \circ f^* = I_K = f^* \circ g^* \Rightarrow f^*$ is a bijective ring homomorphism from K to itself $\therefore$ it is a ring isomorphism. $\square$