MA347 – HW23

Jonathan Lam

April 28, 2021

1. (a) Let I be an ideal of a commutative ring R. Prove that $\text{ann}(I) = \{r \in R : ra = 0\ \forall a \in I\}$ is an ideal of R.

   Proof. Let $x_1, x_2 \in \text{ann}(I)$, $r \in R$. Then, $\forall a \in I$:

   $$a(x_1 + x_2) = ax_1 + ax_2 = 0 + 0 = 0$$

   and

   $$a(rx_1) = r(ax_1) = r(0) = 0$$

   $\therefore x_1 + x_2 \in \text{ann}(I), rx_1 = x_1 r \in \text{ann}(I) \Rightarrow \text{ann}(I)$ is a (two-sided) ideal. $\square$

   (b) Let $R = \mathbb{Z}/20\mathbb{Z}$ and $I = \{[n] \in R : n \text{ is even}\}$. Find $\text{ann}(I)$.
   We use the looser notation $r \equiv [r] \in R$.

   $$\begin{aligned}
   \text{ann}(I) &= \{r \in \mathbb{Z}/20\mathbb{Z} : ra = 0\ \forall a \in I\} \\
   &= \{r \in \mathbb{Z}/20\mathbb{Z} : ra = 0\ \forall a \in \{0, 2, \ldots, 18\}\} \\
   &= \{r \in \mathbb{Z}/20\mathbb{Z} : ra \equiv 0 \pmod{20}\ \forall a \in \{0, 2, \ldots, 18\}\} \\
   &= \{r \in \mathbb{Z}/20\mathbb{Z} : 20 \mid ra\ \forall a \in \{0, 2, \ldots, 18\}\}
   \end{aligned}$$

   Then $\text{ann}(I)$ must be the ideal $\{0, 10\}$, because no other elements have exponent 2 in the additive group, and clearly $0a \equiv 0 \pmod{20}$, $10a \equiv 0 \pmod{20}\ \forall a \in \{0, 2, \ldots, 18\}$.

2. Find all the ideals of a field F.

   The trivial ideal $\{0\} \subseteq F$ is always an ideal.

   Assume an ideal $J \subseteq F$ is non trivial. Let $0 \neq j \in J$. Since F^* is a group under product, $\exists r = j^{-1} \in R$. Since J is an ideal, $1 = j^{-1}j = rj \in R$. Then $\forall r \in R$, $r(1) \in J \Rightarrow R \subseteq J \subseteq R \Rightarrow J = R$.

   $\therefore$ the only two ideals of F are the trivial ideal and F itself.