MA347 – HW21

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1. Let G be a group of order 91.

   (a) Prove that G has only one 7-Sylow subgroup H and only one 13-Sylow subgroup K.

   Proof. Using Sylow's third theorem, there are $m = 1 + 7k$ 7-Sylow subgroups, $m \mid 91$, and there are $n = 1 + 13k$ 13-Sylow subgroups, $n \mid 91$. The only numbers that match these constraints are $m = 1$ and $n = 1$ ($k = 0$ in both cases). $\square$

   (b) Prove that H, K are normal.

   Proof. Let G be a group, and P be the only p-Sylow subgroup. Let $g \in G$. Then the conjugate group gPg^{-1} also is a p-Sylow subgroup ($|gPg^{-1}| = |P|$ since conjugation is a bijection), and must equal P because it is the only p-Sylow subgroup. Thus $gPg^{-1} = P$ for all $g \in G$, so P is normal. (This proof is true more generally for the case of a unique subgroup of any order.)

   Apply this result to H and K to show that they are normal. $\square$

   (c) Prove that $H \cap K = \{e\}$.

   Proof. H and K are both finite groups of prime order, so they are cyclic groups where each non-identity element has period matching the order of the subgroup. Thus no non-identity element can belong to both subgroups (no element can have period both 7 and 13), so the intersection is trivial. $\square$

(d) Prove that $hk = kh \ \forall h \in H, k \in K$, and that H, K are abelian.

Proof. ($hk = kh \ \forall h \in H, k \in K$)
Consider the element $hkh^{-1}k^{-1}$, $h \in H$, $k \in K$. $K \triangleleft G \Rightarrow hkh^{-1} = k_1 \in K$. $H \triangleleft G \Rightarrow kh^{-1}k^{-1} = h_1 \in H$. Then

$$kk_1 = hkh^{-1}k^{-1} = hh_1 \Rightarrow hkh^{-1}k^{-1} \in H \cap K = \{e\}$$

$\therefore hk = kh$.
(H, K abelian) H and K are both finite groups of prime order. Thus it must be a (simple) (cyclic) abelian group (previous HW). □

(e) Prove that G is cyclic and hence abelian.

Proof. Suppose $a = hk \in G$, where $h \in H$, $k \in K$, and $h, k \neq e$. Then h has period 7 and k has period 13. Since H and K are abelian, and since elements of H and K commute with each other, we can write $a^n = (hk)^n = h^n k^n$ (proof using induction is immediate).
We show that 91 is the period of a. It is an exponent of a because $a^{91} = h^{91}k^{91} = (h^7)^{13}(k^{13})^7 = e^{13}e^7 = ee = e$. It is the least positive exponent because for any $0 < n < 91$, 7 and 13 do not both divide n, so h^n and k^n are not both e (nor are they inverses because $H \cap K = \{e\}$), and thus $a^n = h^n k^n \neq e$.
a generates a subgroup of G of order 91, so $G = \langle a \rangle$ and thus abelian. □