MA347 – HW20

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1. Prove that an abelian group A is simple iff it is finite and of prime order.

   Proof. Preliminaries: since A is abelian, every subgroup is normal. If it is simple, every nonzero (non-identity) element $a \in A$ generates a nontrivial subgroup of A, which must be A itself (every nonzero element must be a generator of A).

   ($\Rightarrow$) Assume that an abelian group A is simple and of infinite order. Let $0 \neq a \in A$, and thus a generates A. The nonzero element $2a$ must also generate A, and thus there exists $m \in \mathbb{N}$ such that $m(2a) = a$. Subtracting a from both sides, we have $(2m-1)a = 0$, and thus a has a finite exponent $\Rightarrow A = \langle a \rangle$ is finite $\Rightarrow$ contradiction. $\therefore A$ is simple $\Rightarrow A$ is finite.

   Let A be a finite abelian group with non-prime order, and let p be a prime that divides $|A|$. By Lemma 9.2 (Lang), A has a (proper) subgroup of order p, so A is not simple. By contrapositive, A must have prime order.

   ($\Leftarrow$) By Lagrange's theorem, any subgroup of a (finite) group A must have order dividing A. A has prime order, so any (normal) subgroup must be the trivial group or itself. $\therefore A$ is simple. $\square$

2. Prove that S_4 has more than one 2-Sylow subgroup.

   Proof. By Sylow's third theorem we know that S_4 has either one or three 2-sylow subgroups (subgroups of order 8).

   Consider S_4 as the set of permutations of the integers J_4. Choose the subgroup of S_4 that maintains the "partition" of S_4 $\{1,2\} \cup \{3,4\}$, i.e., the permutations that map $\{1,2\}$ onto either $\{1,2\}$ or $\{3,4\}$:

   $$\{\varepsilon, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\} = H < S_4$$

   This can be shown to be a subgroup (almost by definition). These are all the possibilities that preserve this partition; for $\sigma \in H$, $\sigma(1)$ has four possibilities, $\sigma(2)$ is fixed by $\sigma(1)$, $\sigma(3)$ has two possibilities (limited by the choice of $\sigma(1)$), and $\sigma(4)$ is fixed by the other choices, so $|H| = 8$.

   There are three unique partitions of S_4 into two sets of two elements, so these must be the three 2-Sylow subgroups of S_4. $\square$