MA347 – HW19

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1. Let A be an abelian group where $A = \mathbb{C}^*$ under product. Let p be a prime. Find $A(p)$, the p-primary component of A.

   By definition,

$$\begin{aligned} A(p) &= \left\{ x \in \mathbb{C}^* : \exists n \in \mathbb{Z}^+ \text{ s.t. } x^{(p^n)} = 1 \right\} \\ &= \bigcup_{n=0}^{\infty} \left\{ x \in \mathbb{C}^* : x^{(p^n)} = 1 \right\} \\ &= \bigcup_{n=0}^{\infty} \left\{ \exp \frac{2\pi i k}{p^n} : 0 \leq k < p^n \right\} \end{aligned}$$

   This is the set of the zeroth/first root of 1 (1 itself), p-roots of 1, p^2-roots of 1, p^3-roots of 1, etc. Since each p^n-th root of 1 is also a p^m-th root of 1 if $m > n$, this is the set of "p^∞-th roots of 1."

   (From a Google search, this group is called the Prüfer p-group $Z(p^\infty)$.)

2. Let G be a group and $H \triangleleft G$. Prove that:

   (a) If G is a p-group (for $p \in \mathbb{N}$ prime), then H is a p-group and ${}^G/_H$ is a p-group.

   Proof. By definition, a p-group has (finite) order p^n, $n \in \mathbb{Z}^+$. Then $|G| = p^a$, $a \in \mathbb{Z}^+$. We have:

$$|G| = (G : H)|H| = |{}^G/_H||H|$$

   By the Fundamental Theorem of Arithmetic, p^a can only be the product of the form $p^b p^c$, for $0 \leq b, c \leq a$ and $b + c = a$. Thus H and ${}^G/_H$ (a group because $H \triangleleft G$) are p-groups. □

   (b) If H and ${}^G/_H$ are p-groups, then G is a p-group.

   Proof. H is a p-group $\Rightarrow |H| = p^c$ for $c \in \mathbb{Z}^+$. ${}^G/_H$ is a p-group $\Rightarrow |{}^G/_H| = (G : H) = p^b$ for $b \in \mathbb{Z}^+$. Then

$$|G| = (G : H)|H| = p^b p^c = p^{b+c}$$

   The order of G is a power of p, so G is a p-group. □