MA347 – HW17

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1. Let G be a group and H be a subgroup. Suppose for each $a, b \in G$ there exists $c \in G$ such that $aHbH = cH$. Prove that ${}^{G}\!/_{H}$, the set of left cosets of H in G, is a group.

   Proof. The group operation is a binary (closed) operation by definition, since for all $x, y \in G$, $(xH)(yH) = xHyH = cH \in {}^{G}\!/_{H}$ for some $c \in H$.

   GRP1 Product of (co)sets is always associative. Consider the product $((aH)(bH))(cH)$, where $a, b, c \in G$.

   $$\begin{aligned} ((aH)(bH))(cH) &= \{(a'b')c' : a' \in aH, b' \in bH, c' \in cH\} \\ &= \{a'(b'c') : a' \in aH, b' \in bH, c' \in cH\} \\ &= (aH)((bH)(cH)) \end{aligned}$$

   GRP2 Let $e' = H = eH \in {}^{G}\!/_{H}$, where e is the unit element of G. Then $\forall aH \in {}^{G}\!/_{H}$:

   $$(aH)(H) = a(HH) = aH$$

   Conversely, let $x \in H(aH)$. Then $x = h_1 a h_2$. In particular, let $h_1 = e \Rightarrow x \in aH$. We know that $H(aH)$ is a left coset of H in G. Since left cosets form an equivalence relation over G, $x \in aH \Rightarrow H(aH)$ is precisely the coset aH.

   $(aH)e' = aH = e'(aH) \therefore e' = H$ is the unit element of ${}^{G}\!/_{H}$.

   GRP3 Let $aH \in {}^{G}\!/_{H}$. Then $a^{-1}H \in {}^{G}\!/_{H}$. Then $(aH)(a^{-1}H)$ is a left coset of H in G. Let $x = ah_1a^{-1}h_2 \in (aH)(a^{-1}H)$. In particular, let $h_1 = e$. Then $x = aea^{-1}h_2 = aa^{-1}h_2 \in eH = H = e'$. Since left cosets form an equivalence relation over G, $x \in aH \Rightarrow (aH)(a^{-1}H)$ is precisely the coset $H = e'$.

   The same argument can be used to show that $(a^{-1}H)(aH) = H = e'$. Thus an inverse element exists for each $aH \in {}^{G}\!/_{H}$.

   This set of left cosets with the stated property is closed over the binary group operation and the group axioms are satisfied, so it is a group under product. □

2. Let G and G' be abelian groups and $f : G \to G'$ be a homomorphism. Assume there exists a homomorphism $g : G' \to G$ such that $f \circ g = I_{G'}$.

   (a) Prove that $G = \operatorname{Ker} f \oplus \operatorname{Im} g$.

   Proof. ($G = \operatorname{Ker} f + \operatorname{Im} g$) Let $x \in G$. Then $y = g(f(x)) \in \operatorname{Im} g$. Let $z = x - y$. Then

$$\begin{aligned}
f(z) &= f(x-y) && \text{(def. } z\text{)}\\
&= f(x) - f(y) && (f \text{ homo.})\\
&= f(x) - f(g(f(x))) && \text{(def. } y\text{)}\\
&= f(x) - f(x) && (f \circ g = I_{G'})\\
&= e' \\
&\Rightarrow z \in \operatorname{Ker} f
\end{aligned}$$

   Thus $x = z + y$ for $z \in \operatorname{Ker} f$, $y \in \operatorname{Im} g$. $\therefore G = \operatorname{Ker} f = \operatorname{Im} g$.

   ($\operatorname{Ker} f \cap \operatorname{Im} g = \{0\}$) Let $x \in \operatorname{Ker} f \cap \operatorname{Im} g$. Then $\exists y' \in G'$ such that $y = g(y') \Rightarrow f(y) = g(f(y')) = y' = 0' \Rightarrow y = g(y') = g(0') = 0$. Thus $\operatorname{Ker} f \cap \operatorname{Im} g = \{0\}$.

   $G = \operatorname{Ker} f + \operatorname{Im} g, \operatorname{Ker} f \cap \operatorname{Im} g = \{0\} \Rightarrow G = \operatorname{Ker} f \oplus \operatorname{Im} g$. □

   (b) Prove that f and g are inverse isomorphisms between the abelian groups $g(G')$ and G'.

   Proof. Restrict the domain of f and codomain of g to be the subgroup $g(G') \subseteq G$:

$$\begin{aligned}
&f : g(G') \to G'\\
&g : G' \to g(G')
\end{aligned}$$

   ($g \circ f = I_{g(G')}$) Let $x \in g(G')$. Then $x = g(x')$ for some $x' \in G'$, and

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) && \text{(def. } \circ\text{)}\\
&= g(f(g(x'))) && (x \in g(G'))\\
&= g(x') && (f \circ g = I_{G'})\\
&= x && (x = g(x'))\\
&= I_{g(G')}(x) && \text{(def. } I_{g(G')}\text{)}
\end{aligned}$$

   We have $f : g(G') \to G'$, $g : G' \to g(G')$ homomorphisms, and $f \circ g = I_{G'}$ (given), $g \circ f = I_{g(G')}$ (just proved). $\therefore f$ and g are inverse isomorphisms between G' and $g(G')$. □