MA347 – HW15

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(Problems II.§8.2, II.§8.3a from Lang)

1. Let $\pi : G \to \mathrm{Perm}(S)$ be a homo. where G is a group and S is a set. Prove that $\operatorname{Ker}\pi = \bigcap_{s \in S} G_s$ where G_s is an isotropy group of $s \in S$.

   Proof. Let $K = \operatorname{Ker}\pi, H = \bigcap_{s \in S} G_s$ for convenience.
   ($K \subseteq H$)

   $$\begin{aligned} K &= \{x \in G : \pi(x) = I_S\} \\ &= \{x \in G : xs = s \ \forall s \in S\} \\ &= \{x \in G : x \in G_s \ \forall s \in S\} \\ &\Rightarrow K \subseteq G_s \ \forall s \in S \\ &\Rightarrow K \subseteq H \end{aligned}$$

   ($H \subseteq K$) Assume not. Then $\exists x \in G$ such that $x \in H$ and $x \notin K$. Then

   $$\begin{aligned} x \notin K &\Rightarrow x \notin \{x \in G : \pi(x) = I_S\} = \{x \in G : xs = s \ \forall s \in S\} \\ &\Rightarrow \exists s \in S \text{ such that } xs \neq s \\ &\Rightarrow x \notin G_s \\ &\Rightarrow x \notin H \\ &\Rightarrow \text{contradiction} \Rightarrow H \subseteq K \end{aligned}$$

   $K \subseteq H \subseteq K \Rightarrow K = H.$ □

2. Let G be a group of order p^n where p is prime and $n \in \mathbb{N}$. Prove that $Z(G)$ is nontrivial, i.e., that $|Z(G)| > 1$.

   By the class formula (Lang prop. 8.5), we have:

$$|G| = |Z(G)| + \sum_{i=1}^{m} (G : G_{y_i})$$

   where $\{y_i\}_{i=1}^m$ represent the conjugacy classes which contain more than one element, and $(G : G_{y_i}) > 1$ for $i = \{1, \ldots, m\}$.

   Since $|G| = (G : G_{y_i})|G_{y_i}|$, then $(G : G_{y_i})$ divides $|G| = p^n$, so $(G : G_{y_i}) = p^{r_i}$, where $1 \leq r_i < n$ for all $i = \{1, \ldots, m\}$.

   Using to the property that divisibility distributes over sums, we have:

$$\begin{aligned} & p \mid |G|, \; p \mid (G : G_{y_i}) \; \forall i = \{1, \ldots, m\} \\ \Rightarrow & \, p \mid \left(|G| - \sum_{i=1}^{m} (G : G_{y_i}) = |Z(G)| \right) \end{aligned}$$

   Thus $|Z(G)|$ is a (non-zero) multiple of p.