MA347 – HW14

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Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 13 & 2 & 15 & 14 & 10 & 6 & 12 & 3 & 4 & 1 & 7 & 9 & 5 & 11 & 8 \end{pmatrix}$.

1. Write α as a product of disjoint cycles and find $|\alpha|$ (the period of α).

$$\alpha = \begin{pmatrix} 1 & 13 & 5 & 10 \end{pmatrix} \begin{pmatrix} 3 & 15 & 8 \end{pmatrix} \begin{pmatrix} 4 & 14 & 11 & 7 & 12 & 9 \end{pmatrix} \begin{pmatrix} 2 \end{pmatrix} \begin{pmatrix} 6 \end{pmatrix}$$
$$|\alpha| = \text{lcm}\{4, 3, 6, 1, 1\} = 12$$

(The 1-cycles are decorative as they are equal to ε, the identity map.)

2. Determine the sign of α.

   Preliminary material: Using the notation from Lang, let $\epsilon : S_n \to \{1, -1\}$ denote the map from permutations of J_n to their sign. (-1 denotes an odd permutation, and 1 denotes an even permutation.) As noted in Thm. 6.4 of Lang, this is a group homomorphism.

   Claim: an r-cycle is even iff. r is odd. I.e., $\epsilon(\sigma_r) = (-1)^{r+1}$, where σ_r is a r-cycle (problem II.§6.5 in Lang).

   Proof. (Using induction) Let $\sigma = \begin{pmatrix} i_1 & \dots & i_r \end{pmatrix}$ be an r-cycle. Let $A(n)$ be the assertion that $\epsilon\left(\begin{pmatrix} i_1 & \dots & i_n \end{pmatrix}\right) = (-1)^{n+1}$, for $2 < n \leq r$.

   Base case ($n = 2$): A transposition (2-cycle) is odd. Thus $A(2)$ is true.

   Inductive step ($n > 2$, assume $A(n-1)$ is true): We have:

   $$\begin{pmatrix} i_1 & \cdots & i_n \end{pmatrix} = \begin{pmatrix} i_1 & i_n \end{pmatrix} \begin{pmatrix} i_1 & \cdots & i_{n-1} \end{pmatrix}$$

   Denote these as:

   $$\sigma_n = \tau \sigma_{n-1}$$

   We see this is true because σ_n and σ_{n-1} are almost the same, except that σ_{n-1} fixes n and $\sigma_{n-1}(i_{n-1}) = i_1$, while σ_n does maps $\sigma_n(i_{n-1}) = i_n$ and $\sigma_n(i_n) = i_1$. By composing τ with σ_{n-1}, we now have $(\tau\sigma_{n-1})(i_{n-1}) = \tau(i_1) = i_n$ and $(\tau\sigma_{n-1})(i_n) = \tau(i_n) = i_1$ while everything else is left unchanged, so $\tau\sigma_{n-1} = \sigma_n$.

   Then by inductive hypothesis, $\epsilon(\sigma_{n-1}) = (-1)^(n - 1 + 1) = (-1)^n$, and:

   $$\begin{aligned} \epsilon(\sigma_n) &= \epsilon(\tau\sigma_{n-1}) \\ &= \epsilon(\tau)\epsilon(\sigma_{n-1}) && (\epsilon \text{ is homo.}) \\ &= (-1)\epsilon(\sigma_{n-1}) && (\text{transposition is odd}) \\ &= (-1)(-1)^n && (\text{by inductive hyp.}) \\ &= (-1)^{n+1} \\ &\therefore A(n-1) \Rightarrow A(n) \end{aligned}$$

   By first form of induction, $A(n)$ is true for all $n \in \mathbb{N}$ for $2 \leq n \leq r$. In particular having $A(r)$ be true proves the claim. □

   To find the sign of α, use the fact that ϵ is a homomorphism to decompose the sign into the product of the signs of the component cycles, and then use the above result to calculate the sign of each of the component r-cycles.

   $$\begin{aligned} \epsilon(\alpha) &= \epsilon\left(\begin{pmatrix} 1 & 13 & 5 & 10 \end{pmatrix}\begin{pmatrix} 3 & 15 & 8 \end{pmatrix}\begin{pmatrix} 4 & 14 & 11 & 7 & 12 & 9 \end{pmatrix}\right) \\ &= \epsilon\left(\begin{pmatrix} 1 & 13 & 5 & 10 \end{pmatrix}\right)\epsilon\left(\begin{pmatrix} 3 & 15 & 8 \end{pmatrix}\right)\epsilon\left(\begin{pmatrix} 4 & 14 & 11 & 7 & 12 & 9 \end{pmatrix}\right) \\ &= (-1)(1)(-1) = 1 \\ &\therefore \alpha \text{ is even} \end{aligned}$$