MA347 – HW13

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1. N and M are normal subgroups of a group G. Prove that NM is a normal subgroup of G.

   Proof. Let $a, b \in NM$. Then $a = m_1 n_1$, $b = m_2 n_2$ for some $m_1, m_2 \in M$, $n_1, n_2 \in N$. Thus

$$\begin{aligned} ab^{-1} &= (n_1 m_1)(n_2 m_2)^{-1} \\ &= n_1 m_1 m_2^{-1} n_2^{-1} \\ &= n_1 m_3 n_2^{-1} \\ &= n_1 (m_3 n_2^{-1}) \end{aligned}$$

   where $m_3 = m_1 m_2^{-1} \in M$. Since $M \triangleleft G$, then $m_3 n_2^{-1} \in M n_2^{-1} = n_2^{-1} M \Rightarrow \exists m_4 \in M$ such that $m_3 n_2^{-1} = n_2^{-1} m_4 \in M$. Substituting:

$$\begin{aligned} ab^{-1} &= n_1 (m_3 n_2^{-1}) \\ &= n_1 (n_2^{-1} m_4) \\ &= (n_1 n_2^{-1}) m_4 \\ &= n_3 m_4 \in NM \end{aligned}$$

   where $n_3 = n_1 n_2^{-1} \in N$. $\forall a, b \in NM$ $ab^{-1} \in NM \Rightarrow NM \leq G$.

   To show $NM \triangleleft G$, we show that $x(NH) = (NH)x$ $\forall x \in G$ (**NOR1**).

   Suppose $x n_1 m_1 \in x(NH)$, where $x \in G$, $n_1 \in N$, $m_1 \in M$. Using the property that $N \triangleleft G$. $x n_1 \in xN = Nx \Rightarrow \exists n_2 \in N$ such that $x n_1 = n_2 x \in Nx$. Similarly, $\exists m_2 \in M$ such that $x m_1 = m_2 x \in M_x$. Thus:

$$\begin{aligned} & x(n_1 m_1) \in x(NM) \\ &= (x n_1) m_1 \\ &= (n_2 x) m_1 \\ &= n_2 (x m_1) \\ &= n_2 (m_2 x) \\ &= (n_2 m_2) x \in (NM)x \end{aligned}$$

   Thus $x(NM) \subseteq (NM)x$ $\forall x \in G$. Similarly, we can show that $(NM)x \subseteq x(NH) \Rightarrow (NM)x = x(NM)$ $\forall x \in G$. $\quad \therefore NM \triangleleft G$ $\square$

2. Let G be a group and let S be the set of subgroups of G. Define, for H and K in S, $H \underset{c}{\sim} K$ iff. $\exists x \in G$ s.t. $xHx^{-1} = K$. (We say that H is conjugate to K if $H \underset{c}{\sim} K$.)

   (a) Prove that $\underset{c}{\sim}$ is an equivalence relation in S.

   Proof. **Reflexivity** Let $H \in S$. Then $\exists x = e \in G$ s.t.

$$\begin{aligned} xHx^{-1} &= eHe^{-1} \\ &= (eH)e \\ &= He && (e \in H,\ hH = H\ \forall h \in H) \\ &= H && (e \in H,\ Hh = H\ \forall h \in H) \end{aligned}$$

   $\therefore H \underset{c}{\sim} H$.

   Symmetry Let $H, K \in S$, $H \underset{c}{\sim} K$. Then $\exists x \in G$ s.t. $xHx^{-1} = K$. Then it is clear that $H = x^{-1}Kx = yKy^{-1}$. Thus $\exists y = x^{-1} \in G$ s.t. $yKy^{-1} = H \Rightarrow K \underset{c}{\sim} H$.

   Transitivity Let $H, K, L \in S$, and $H \underset{c}{\sim} K$, $K \underset{c}{\sim} L$. Then $\exists x, y \in G$ s.t. $xHx^{-1} = K$ and $yKy^{-1} = L$. Then:

$$\begin{aligned} L &= yKy^{-1} \\ &= y(xHx^{-1})y^{-1} \\ &= (yx)H(x^{-1}y^{-1}) && \text{(See below comment)} \\ &= (yx)H(yx)^{-1} \end{aligned}$$

   (Comment: It is intuitive that $y(xHx^{-1})y^{-1} = (yx)H(x^{-1}y^{-1})$, but to explicitly show that this is not simply an abuse of notation: this holds true because each element is of the form $yxhy^{-1}x^{-1}$ where $h \in H$, which belongs to both sets.)

   Thus $\exists z = yx \in G$ s.t. $zHz^{-1} = L \Rightarrow H \underset{c}{\sim} L$.

   $\underset{c}{\sim}$ is reflexive, symmetry, and transitive $\therefore$ equivalence relation. □

   (b) Find the equivalence class $[H]_c$ of $H \in S$.

   By definition, $[H]_c = \{K \in S : \exists x \in G \text{ s.t. } xHx^{-1} = K\}$. Another way to describe each $K \in [H]_c$ is that $K = c_x(H)$, i.e., the image of H under conjugation by x (for some $x \in G$). Since c_x is a bijective homomorphism, then $H \underset{c}{\sim} K$ implies that $H \simeq K$, so the equivalence class includes all subgroups of G isomorphic to H under the conjugation map.