MA347 – HW12

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1. Let G be a group. $a \in G$, $c_a : G \to G$ defined by $c_a(x) = axa^{-1}$. Then $c_a \in \mathrm{Aut}(G)$. $\mathrm{Inn}(G) = \{e_a : a \in G\}$ is a subgroup of G and $\varphi : G \to \mathrm{Aut}(G)$ defined by $\varphi(a) = c_a$ is a homo. Prove that:

   (a) $Z(G) = \mathrm{Ker}\,\varphi$ where $Z(G)$ is the center of φ.

   Proof.

   $$\begin{aligned}\mathrm{Ker}\,\varphi &= \{a \in G : \varphi(a) = e'\}\\ &= \{a \in G : c_a = I_G\}\\ &= \{a \in G : c_a(x) = I_G(x)\ \forall x \in G\}\\ &= \{a \in G : axa^{-1} = x\ \forall x \in G\}\\ &= \{a \in G : ax = xa\ \forall x \in G\}\\ &= Z(G)\end{aligned}$$

   $\square$

   (b) $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$. ($\mathrm{Out}(G) := {}^{\mathrm{Aut}(G)}/_{\mathrm{Inn}(G)}$ is called the outer automorphism group.)

   Proof. Let $K = \mathrm{Inn}(G)$, $H = \mathrm{Aut}(G)$. We show the result by **NOR1**, i.e., that $xKx = K\ \forall x \in H$.

   Let $h = c_a \in \mathrm{Inn}\,G$. $x \circ h \circ x \in xHx^{-1}$. Then, $\forall y \in G$:

   $$\begin{aligned}(x \circ h \circ x^{-1})(y) &= (x \circ h)(x^{-1}(y))\\ &= x(ax^{-1}(y)a^{-1})\\ &= x(a)x(x^{-1}(y))x(a^{-1}) && (x \text{ is homo.})\\ &= x(a)I_G(y)x(a^{-1}) && (\text{def. } x^{-1})\\ &= (x(a))y(x(a))^{-1} && (x \text{ is homo.})\\ &= c_{x(a)}(y)\end{aligned}$$

   thus $x \circ h \circ x \in H \Rightarrow xHx^{-1} \subseteq H$. Since $x^{-1} \in G$, $x^{-1}Hx \subseteq H \Rightarrow H \subseteq xHx^{-1}$.
   $\therefore xHx^{-1} = H$. $\square$

2. Let $H \leq G$. Define $N_H \coloneqq N(H) = \{x \in G : xHx^{-1} = H\}$. Prove that:

   (a) xHx^{-1} is a subgroup of G.

   Proof. Let $a, b \in xHx^{-1} \subseteq G$. Then $a = xh_1x^{-1}$, $b = xh_2x^{-1}$ for some $h_1, h_2 \in H$, and

$$\begin{aligned} ab^{-1} &= (xh_1x^{-1})(xh_2x^{-1})^{-1} \\ &= (xh_1x^{-1})(xh_2^{-1}x^{-1}) \\ &= xh_1(x^{-1}x)h_2^{-1}x^{-1} \\ &= xh_1h_2^{-1}x^{-1} \end{aligned}$$

   Since H is a group and thus closed under the group operation and its inverse, $h_1h_2^{-1} = h_3 \in H$, and $ab^{-1} = xh_3x^{-1} \in xHx^{-1}$.

   $\therefore xHx^{-1} \leq G$. $\square$

   (b) $N_H = N(H)$ is a subgroup of G and H is a normal subgroup of $N(H) = N_H$.

   Proof. ($N_H \leq G$) Let $a, b \in N_H \subseteq G$. Clearly $b^{-1} \in N_H$ as well since $bHb^{-1} = H \Leftrightarrow H = b^{-1}Hb$. Then $aH = Ha$ and $b^{-1}H = Hb^{-1}$, and

$$\begin{aligned} (ab^{-1})H &= (aH)(b^{-1}H) && \text{(def. product of normal subgroups)} \\ &= (Ha)(Hb^{-1}) && (a, b^{-1} \in N_H) \\ &= H(ab^{-1}) && \text{(def. product of normal subgroups)} \end{aligned}$$

   Thus $(ab^{-1})H(ab^{-1})^{-1} = H \Rightarrow ab^{-1} \in N_H$.

   $\therefore N_H \leq G$. $\square$

   Proof. ($H \lhd N_H$) H is a group by hypothesis. If $h \in H \subseteq G$, then $hHh^{-1} = (hH)h^{-1} = Hh^{-1} = H \Rightarrow h \in N_H \Rightarrow H \subseteq N_H$. Thus $H \leq N_H$.

   H is normal in N_H by definition of N_H, since $\forall x \in N_H \subseteq G$, $xHx^{-1} = H$ (**NOR1**). $\square$

   (c) $H \lhd G \Leftrightarrow N_H = G$.

   Proof. By **NOR1**, we have $N_H \lhd G \Leftrightarrow xHx^{-1} = H \ \forall x \in G$. This proves the claim both ways.

   ($\Rightarrow$) Let $H \lhd G$. Then we have $xHx^{-1} = x \ \forall x \in G$. Thus the condition for N_H is satisfied $\forall x \in G$, so $N_H = G$.

   ($\Leftarrow$) Let $N_H = G$. Then $xHx^{-1} = H \ \forall x \in N_H = G \Rightarrow H \lhd G$. $\square$