MA347 – HW11

Jonathan Lam

March 1, 2021

1. Let $\varphi_1 : \text{Aff}(n, \mathbb{K}) \to \text{GL}(n, \mathbb{K})$ be defined by $\varphi_1(f_{A,a}) = A$. Prove that φ_1 is a group homomorphism and find $\operatorname{Ker} \varphi_1$.

   Proof φ_1 homo. Let $f_1 = f_{A_1,a_1}, f_2 = f_{A_2,a_2} \in \text{Aff}(n, \mathbb{K})$. Then, $\forall x \in \mathbb{K}^n$:

$$
\begin{aligned}
(f_1 \circ f_2)(x) &= f_1(f_2(x)) \\
&= f_1(A_2x + a_2) \\
&= A_1(A_2x + a_2) + a_1 \\
&= (A_1(A_2x) + A_1a_2) + a_1 && (L_{A_1} : \mathbb{K}^n \to \mathbb{K}^n \text{ is homo.}) \\
&= A_1(A_2x) + (A_1a_2 + a_1) && (\text{associativity of } \mathbb{K}^n) \\
&= (A_1A_2)x + (A_1a_2 + a_1) && (\text{associativity of GL}(N, \mathbb{K})) \\
&= f_{A_1A_2, A_1a_2 + a_1}
\end{aligned}
$$

   Thus $\varphi_1(f_1 \circ f_2) = \varphi_1(f_{A_1A_2,A_1a_2+a_1}) = A_1A_2 = \varphi_1(f_1)\varphi_1(f_2)$.
   $\therefore \varphi_1$ is a group homomorphism. □

   Finding the kernel of φ_1:

$$
\begin{aligned}
f_{A,a} \in \operatorname{Ker} \varphi_1 &\Rightarrow \varphi_1(f_{A,a}) = e_{\text{GL}(n,\mathbb{K})} \\
&\Rightarrow A = I_n \\
&\Rightarrow \operatorname{Ker} \varphi_1 = \{f_{I_n,a} : a \in \mathbb{K}^n\}
\end{aligned}
$$