MA347 – HW10

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Let $H \leq G$. Define $a \underset{H}{\sim} b$ iff. $b^{-1}a \in H$.

1. Prove that $\underset{H}{\sim}$ is an equivalence relation on G and $[a] = aH$, the left coset of H by a.

   Proof equivalence relation. Fix $H \leq G$.

   Symmetry Let $a, b \in G$ and $a \underset{H}{\sim} b$. Then

   $$\begin{aligned} a \underset{H}{\sim} b &\Rightarrow b^{-1}a \in H && (\text{def. } \underset{H}{\sim}) \\ &\Rightarrow (b^{-1}a)^{-1} = a^{-1}b \in H && (\text{GRP 3}) \\ &\Rightarrow b \underset{H}{\sim} a && (\text{def. } \underset{H}{\sim}) \end{aligned}$$

   Reflexivity Let $a \in G$. Then $a^{-1}a = e \in H$ (GRP 1).

   Transitivity Let $a, b, c \in G$, and $a \underset{H}{\sim} b, b \underset{H}{\sim} c$. Then:

   $$\begin{aligned} a \underset{H}{\sim} b &\Rightarrow b^{-1}a \in H \\ b \underset{H}{\sim} c &\Rightarrow c^{-1}b \in H \\ &\Rightarrow (c^{-1}b)(b^{-1}a) \in H && (\text{group closed over multiplication}) \\ &\Rightarrow c^{-1}(bb^{-1})a \in H && (\text{group operation is associative}) \\ &\Rightarrow c^{-1}ea = c^{-1}a \in H \\ &\Rightarrow a \underset{H}{\sim} c \end{aligned}$$

   The relation $\underset{H}{\sim}$ is symmetric, reflexive, transitive $\therefore$ equivalence relation.

   □

Proof equivalence classes. We wish to show that

$$[a] = \{x \in G : a \underset{H}{\sim} x \Leftrightarrow x^{-1}a \in H\} = \{ah : h \in H\} = aH$$

Claim: $aH \subseteq [a]$. Proof: Let $x \in aH \Rightarrow x = ah$ for some $h \in H$. Then

$$\begin{aligned} a(a^{-1}h) &= (aa^{-1})h \\ &= eh = h \in H \\ &\Rightarrow x \underset{H}{\sim} a \Rightarrow a \underset{H}{\sim} x \Rightarrow x \in [a] \end{aligned}$$

Claim: $[a] \subseteq aH$. Proof: Let $x \in [a]$. Then

$$\begin{aligned} x \in [a] &\Rightarrow x \underset{H}{\sim} a \\ &\Rightarrow a \underset{H}{\sim} x && (\underset{H}{\sim} \text{ symmetric}) \\ &\Rightarrow a^{-1}x = h \in H \\ &\Rightarrow a(a^{-1}x) = ah,\ h \in H \\ &\Rightarrow x = ah,\ h \in H \\ &\Rightarrow x \in aH \end{aligned}$$

Thus we have $aH \subseteq [a] \subseteq aH \Rightarrow aH = [a]$. $\square$

2. Let $S/\sim$ be the set of equivalence classes w.r.t. $\underset{H}{\sim}$. Find $S/\sim$.

   We found the set of equivalence classes of S w.r.t. $\underset{H}{\sim}$ in the previous question. Namely, this is the set of left cosets of H in G, i.e., $S/\sim = G/H$.