MA347 – HW1

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For these questions, we assume that $\mathbb{N}$ and $\mathbb{Z}$ are closed over addition and multiplication, and multiplication distributes over addition. and define $-\mathbb{N} := \{-n : n \in \mathbb{N}\}$, just like we did in class.

1. Prove that there is no integer between 0 and 1.

   Lemma. *Let $a \in \mathbb{N}$, $b \in -\mathbb{N}$. Then $ab \in -\mathbb{N}$. (Or, more simply, if $a > 0$, $b < 0$, then $ab < 0$.)*

   Proof of lemma. Let $c = -b \in \mathbb{N}$. Then $ac \in \mathbb{N}$. Then:

   $$ab + ac = a(b + c) = a(b + (-b)) = a(0) = 0 \Rightarrow 0 - ab = ac \in \mathbb{N}$$

   Thus $ab < 0$. □

   Proof. Let $S = \{s \in \mathbb{Z} : 0 < s < 1\}$, and assume that S is nonempty (i.e., that there exists an integer between 0 and 1). Since S consists only of positive integers, $S \subseteq \mathbb{N}$. By WOP there exists a least element $n_0 \in S$.

   Now, let us examine $n_0^2 = n_0 n_0$. Since $\mathbb{N}$ is closed over multiplication, $n_0^2 \in \mathbb{N}$, and $n_0 < 1 \Rightarrow n_0 - 1 < 0$:

   $$n_0^2 - n_0 = n_0(n_0 - 1) < 0 = n_0 - n_0$$

   by the lemma above. Adding n_0 to both sides, we get $n_0^2 < n_0$. However, this contradicts the assertion that n_0 is a least element in $\mathbb{N}$, and thus S must be empty. Thus there exist no integers that lie between 0 and 1. □

2. (Page 5 no. 4) Prove

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^{k}=\frac{(n+1)^{n}}{n!}$$

Proof. Let the hypothesis be called $A(n)$. The base case is $n = 1$:

$$\left(1+\frac{1}{1}\right)^{1}=2=\frac{(1+1)^{1}}{1!}$$

and thus $A(1)$ is true. Inductive step:

$$\begin{aligned}\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)^{i} &= \left[\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)^{i}\right]\left(1+\frac{1}{k+1}\right)^{k+1} \\ &= \left(\frac{(k+1)^{k}}{k!}\right)\left(\frac{k+2}{k+1}\right)^{k+1} \\ &= \frac{(k+2)^{k+1}}{(k+1)k!} \\ &= \frac{((k+1)+1)^{(k+1)}}{(k+1)!}\end{aligned}$$

and thus $A(k)$ is true implies that $A(k+1)$ is true. By induction first form, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$