ECE302 – Quiz 5

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A random variable R is observed, and it is known that under H_0:

$$f_0(r) = f_{R|H_0}(r \mid H_0) = \begin{cases} \frac{1}{2}, & -1 \le r \le 1 \\ 0, & \text{else} \end{cases}$$

and under H_1:

$$f_1(r) = f_{R|H_1}(r \mid H_1) = \frac{1}{2}e^{-|r|}$$

Furthermore, assume the priors $p_0 = \frac{1}{3}$ and $p_1 = \frac{2}{3}$.

1. Plot the class conditional p.d.f.'s on one axis. Be sure to label.

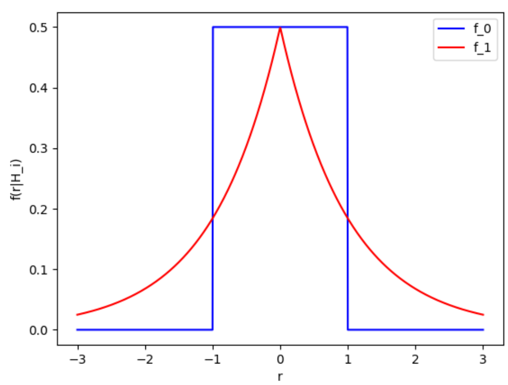

Figure 1: Plot of class conditional densities on the interval $-3 \le r \le 3$

2. Consider a decision rule: If $|r| > \gamma$ decide H_0 ($\hat{H} = 0$), else decide H_1 ($\hat{H} = 1$). Determine the probability of error for this rule if $\gamma = \frac{1}{2}$.

   Using the symmetry of the problem, we can focus on the positive half-line. The decision rule then becomes: if $0 \leq r \leq \frac{1}{2}$, then choose H_1; if $r > \frac{1}{2}$ choose H_0. Thus:

$$\begin{aligned}
P_{err} &= P(\hat{H} = 1 \cap H_0) + P(\hat{H} = 0 \cap H_1) \\
&= \left[\int_0^{\frac{1}{2}} f(r \mid H_0)\, dr\right] \left(\frac{1}{3}\right) + \left[\int_{\frac{1}{2}}^{\infty} f(r \mid H_1)\, dr\right] \left(\frac{2}{3}\right) \\
&= \left[\int_0^{\frac{1}{2}} \frac{1}{2}\, dr\right] \left(\frac{1}{3}\right) + \left[\int_{\frac{1}{2}}^{\infty} \frac{1}{2} e^{-r}\, dr\right] \left(\frac{2}{3}\right) \\
&= \frac{1}{12} + \frac{1}{3} e^{-\frac{1}{2}} \approx 0.286
\end{aligned}$$

3. Determine the likelihood ratio test that minimizes the overall probability of error.

   Again, we focus only on the positive half-line because of the symmetry. The MAP rule states that the decision boundary should occur when:

$$f(r \mid H_1) p_1 = f(r \mid H_0) p_0$$

   In other words, choose the decision boundary η such that:

$$\begin{aligned}
\left(\frac{1}{2} e^{-\eta}\right)\left(\frac{2}{3}\right) &= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \\
e^{-\eta} &= \frac{1}{2} \\
\eta &= \ln 2
\end{aligned}$$

   Thus the MAP rule is: for $|r| > \eta$, choose H_0; otherwise choose H_1.

4. Now suppose p_0 and p_1 are no longer constrained, but cannot be either 0 or 1. Find which values of p_0 such that the decision rule that minimizes the probability of error always decides the same hypothesis, regardless of the observation.

   This situation occurs if the a posteriori probabilities (class conditional multiplied by the priors) don't ever overlap; i.e., the a posteriori probability of H_0 is always greater than H_1, or vice versa.

   It is not possible that the a posteriori probability of H_0 is always higher than H_1, because it is 0 for parts of the domain where H_1 is not. The opposite is true, if the a posteriori probability of H_1 is always greater than the a posteriori probability of H_0 for $|r| \leq 1$, and particularly at $|r| = 1$.

$$\left[\frac{1}{2} e^{-1}\right](1 - p_0) > \frac{1}{2} p_0 \Rightarrow p_0 < \frac{1}{e + 1}$$

   The MAP decision rule always chooses H_1 if $p_0 < (e + 1)^{-1}$.