ECE302 – Quiz 4

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Let X be a Pareto R.V. with parameters α and x_m. Both α and x_m are positive, and x_m is the minimum possible value that X can take. The p.d.f. of a Pareto R.V. is as follows:

$$f_X(x) = \begin{cases} \frac{\alpha x_m^\alpha}{x^{\alpha+1}} & x \geq x_m \\ 0 & x < x_m \end{cases}$$

1. Find the M.L. estimate of α from a sequence of observations $\mathbf{X} = \{X_1, \ldots, X_n\}$, assuming x_m is known.

   Assume observations of $\mathbf{X}$ are I.I.D.

$$\begin{aligned} l(\mathbf{X} \mid \alpha, x_m) &= \prod_{j=1}^{n} f_X(X_j \mid \alpha) = \prod_{j=1}^{n} \frac{\alpha x_m^\alpha}{X_j^{\alpha+1}} \\ L(\mathbf{X} \mid \alpha, x_m) &= \ln l(\mathbf{X} \mid \alpha, x_m) \\ &= \sum_{j=1}^{n} \ln \alpha + \alpha \ln x_m - (\alpha+1) \ln X_j \\ \frac{\partial L}{\partial \alpha} &= \sum_{j=1}^{n} \frac{1}{\alpha} + \ln x_m - \ln X_j \end{aligned}$$

   Maximize log-likelihood function by setting the derivative to zero.

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= 0 \\ n\left(\frac{1}{\alpha} + \ln x_m\right) &= \sum_{j=1}^{n} \ln X_j \\ \alpha &= \left(\frac{1}{n}\sum_{j=1}^{n} \ln X_j - \ln x_m\right)^{-1} \end{aligned}$$

2. Argue that if x_m is unknown, the M.L. estimate of this parameter is the minimum observed value of $\mathbf{X}$.

   x_m is bounded by above by $\min \mathbf{X}$ by definition of the Pareto R.V. We also see that the (log) likelihood function increases monotonically as a function of x_m. Thus the value of x_m that maximizes the likelihood function is the its maximum possible value, i.e., $\min \mathbf{X}$.

3. How does this change your answer to part (1)?

   If we know neither α nor x_m, then the M.L. estimate for α becomes

$$\alpha = \left(\frac{1}{n} \sum_{j=1}^{n} \ln X_j - \ln \min \mathbf{X} \right)^{-1}$$