Stochastics
Quiz 3, Spring 2021
Name:

Suppose *X* and *Y* are related by the following joint pdf:

$$p_{X,Y}(x,y) = \begin{cases} 10x & 0 \leq x \leq y^2, 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

a) Find the MMSE estimate of *X* based on *Y*.
b) Find the corresponding value of the mean squared error of this estimate.
c) Find the linear MMSE estimate of *X* based on *Y*.
d) Find the corresponding value of the mean squared the linear estimate.

ECE302

QUIZ 3

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a) $\hat{X}_{MMSE}(Y) = \frac{2}{3}Y^2$

b) $MSE_{\hat{X}_{MMSE}} = \frac{5}{162}$

c) $\hat{X}_{LMMSE}(Y) = -\frac{5}{14} + Y$

d) $MSE_{\hat{X}_{LMMSE}} = \frac{55}{1764}$

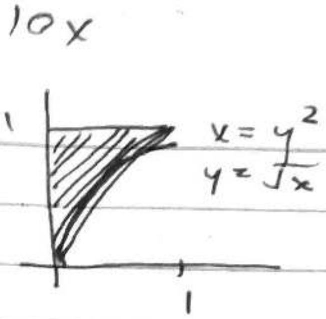
note that $MSE_{\hat{X}_{MMSE}} \approx 0.0309$

is only slightly smaller than $MSE_{\hat{X}_{LMMSE}} \approx 0.0311$

which shows that linear estimate is quite good

$$\hat{y}_{MMSE}(X) = E[Y|X]$$

$$= \int_0^1 y\, f(y|x)\, dy$$



$$f_{X,Y}(x,y) = \begin{cases} 10x, & 0 \le x \le y^2, \quad 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_X(x) = \cancel{\int_0^1 f(x,y)\, dy}$$

$$= \cancel{\int_0^1 10x\, dy = 10xy\Big|_{y=0}^{1}} = 10x$$

$$f_X(x) = \int_{\sqrt{x}}^1 10x\, dx$$

$$= 10xy\Big|_{\sqrt{x}}^{1} = 10x(1-\sqrt{x})$$

$$f_Y(y) = \int_0^{y^2} 10x\, dx$$

$$= \frac{5 \cdot \cancel{10}x^2}{\cancel{2}}\Big|_0^{y^2} = 5y^4.$$

$$\cancel{f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f(x)} = \frac{10x}{10x(1-\sqrt{x})} = \frac{1}{1-\sqrt{x}}}$$

$$\cancel{E[Y|X] = \int_{\sqrt{x}}^1 y\, f(y|x)\, dy}$$

$$\cancel{= \int_{\sqrt{x}}^1 y \frac{1}{1-\sqrt{x}}\, dy}$$

$$= \frac{1}{1-\sqrt{x}} \quad \frac{y^2}{2}\Big|_{\sqrt{x}}^{1}$$

$$= \frac{1}{1-\sqrt{x}}\left(\frac{1}{2} - \frac{x}{2}\right)$$

$$= \frac{1-x}{2(1-\sqrt{x})} = \frac{1}{2}\left(\frac{1^2-\sqrt{x}^2}{1-\sqrt{x}}\right)$$

$$= \frac{1}{2}\,\frac{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}{\cancel{1-\sqrt{x}}}$$

$$= \frac{1+\sqrt{x}}{2}.$$

$$E\left[\left(\hat{Y}_{MMSE}(X) - Y\right)^2\right] = E\left[\left(\frac{1+\sqrt{X}}{2} - Y\right)^2\right]$$

$$= E\left[\left(\frac{1+\sqrt{X}}{2}\right)^2\right.$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{10x}{5y^4} = \frac{2x}{y^4}$$

$$E[X|y] = \int_0^{y^2} x\, f(x|y)\, dx$$

$$= \int_0^{y^2} x\left(\frac{2x}{y^4}\right) dx = \frac{1}{y^4}\int_0^{y^2} 2x^2\, dx$$

$$= \frac{1}{y^4}\,\frac{2x^3}{3}\Big|_0^{y^2}$$

$$= \frac{1}{y^4}\,\frac{2y^6}{3} = \frac{2}{3}y^2.$$

$\hat{X}_{mmse}(Y) = E[X \mid Y] = \frac{2}{3}Y^2$

$E[(\hat{X}_{mmse}(Y) - X)^2]$

$= E[(\frac{2}{3}Y^2 - X)^2]$

$= E[\frac{4}{9}Y^4 - 2(\frac{2}{3}Y^2X) + X^2]$

$= \frac{4}{9}E[Y^4] - \frac{4}{3}E[Y^2X] + E[X^2]$

$= \frac{4}{9}\int_0^1 y^4 f_Y(y)\,dy - \frac{4}{3}\int_0^1\int_0^{y^2} y^2x f_{XY}(x,y)\,dx\,dy$

$+ \int_0^1 x^2 f_X(x)\,dx$

$= \frac{4}{9}\int_0^1 y^4(5y^4)\,dy - \frac{4}{3}\int_0^1\int_0^{y^2} y^2x(10x)\,dx\,dy$

$+ \int_0^1 x^2(10x)(1-\sqrt{x})\,dx$

$\int 5y^8\,dy$

$\Rightarrow \frac{5y^9}{9}\Big|_0^1$

$= \frac{5}{9}$

$\frac{4}{9} \cdot \frac{5}{9} = \frac{20}{81}$

$\int_0^1 10x^3 - 10x^{3.5}\,dx$

$= \frac{10x^4}{4} - \frac{10x^{4.5}}{4.5}\Big|_0^1$

$= \frac{10}{4} - \frac{10}{4.5}$

$= \frac{5}{2} - \frac{20}{9}$

$= \frac{45}{18} - \frac{40}{18} = \frac{5}{18}$

$$\frac{4}{3} \cdot 10 \int_0^1 \int_0^{y^2} y^2 x^2 \, dx \, dy$$

$$= 10\frac{4}{3} \int_0^1 y^2 \int_0^{y^2} x^2 \, dx \, dy$$

$$= 10\frac{4}{3} \int_0^1 y^2 \left. \frac{x^3}{3} \right|_0^{y^2} dy$$

$$= 10\frac{4}{3} \int_0^1 y^2 \frac{y^6}{3} dy$$

$$= \frac{40}{9} \int_0^1 y^8 dy$$

$$= \frac{40}{9} \cdot \left. \frac{y^9}{9} \right|_0^1$$

$$= \frac{40}{81}$$

$$= \frac{20}{81} - \frac{40}{81} + \frac{5}{18}$$

$$= \frac{-20}{81} + \frac{5}{18} = \frac{-40}{162} + \frac{45}{162}$$

$$= \frac{5}{162}$$

LMMSE estimate;

$$Y_{LMMSE}(X) = a_0 + a_1 X$$

$$a_0 = E[Y] - a_1 E[X]$$

$$a_1 = \frac{\sigma_{XY}}{\sigma_X^2} = \frac{\rho_{XY} \sigma_X \sigma_Y}{\sigma_X^2} = \rho_{XY} \frac{\sigma_Y}{\sigma_X}$$

we are estimating X based on Y,
so have to flip this.

Need to find:

σ^2_x , σ^2_y , σ_{xy}, μ_x, μ_y.

$$\mu_x = \int_0^1 x f_x(x)\,dx$$

$$= \int_0^1 x\left(10x\left(1-\sqrt{x}\right)\right)dx$$

$$= \int_0^1 10x^2 - 10x^{2.5}\,dx$$

$$= \frac{10x^3}{3} - \frac{10x^{3.5}}{3.5}\Big|_0^1$$

$$= \frac{10}{3} - \frac{10}{3.5} = \frac{10}{3} - \frac{20}{7}$$

$$= \frac{70}{21} - \frac{60}{21} = \frac{10}{21}.$$

$$\mu_y = \int_0^1 y f_y(y)\,dy$$

$$= \int_0^1 y(5y^4)\,dy$$

$$= \int_0^1 5y^5\,dy = \frac{5}{6}.$$

$$E[x^2] = \frac{5}{18}.$$

$$E[y^2] = \int_0^1 y^2 f_y(y)\,dy$$

$$= \int_0^1 y^2(5y^4)\,dy$$

$$= \int_0^1 5y^6\,dy = \frac{5}{7}$$

~~μ_x~~ $\sigma^2_x = E[x^2] - \mu_x^2$

$= \frac{5}{18} - \left(\frac{10}{21}\right)^2$

$= \frac{5}{18} - \frac{100}{441}$

$2 \cdot 3^2$ $\quad$ $7^2 3^2$

$$\begin{array}{r} 21 \\ \cdot 21 \\ \hline 21 \\ 420 \\ \hline 441 \end{array}$$

$\frac{5 \cdot 49}{882} - \frac{200}{882} = \frac{245 - 200}{882}$

$= \frac{45}{882} = \frac{5}{98}$

$\sigma^2_y = E[y^2] - \mu_y^2$.

$= \frac{5}{7} - \left(\frac{5}{6}\right)^2 = \frac{5}{7} - \frac{25}{36}$.

$$\begin{array}{r} 36 \\ \cdot 7 \\ \hline 252 \end{array}$$

$\frac{180 - 175}{252} = \frac{5}{252}$.

$$\begin{array}{r} 36 \\ \cdot 5 \\ \hline 180 \end{array} \qquad \begin{array}{r} 25 \\ \cdot 7 \\ \hline 175 \end{array}$$

$\sigma_{xy} = E[XY] - \mu_x \mu_y$

$E[xy] = \int_0^1 \int_0^{y^2} xy(10x)\,dx\,dy$

$= 10 \int_0^1 y \int_0^{y^2} \cancel{10}x^2\,dx\,dy$

$= 10 \int_0^1 y \left. \frac{x^3}{3} \right|_0^{y^2} dy$

$= \frac{10}{3} \int_0^1 y \frac{y^6}{\cancel{3}}\,dy$

$= \frac{10}{3} \left. \left(\frac{y^8}{8}\right) \right|_0^1 = \frac{10}{24} = \frac{5}{12}$.

$\sigma_{xy} = \frac{5}{12} - \cancel{\sigma_x \sigma_y}\ \mu_x \mu_y$

$= \frac{5}{12} - \frac{10}{21}\left(\frac{5}{63}\right)$

$= \frac{5}{12} - \frac{25}{63}$

$3 \cdot 2^2 \qquad 3^2 7$

12
7
84
3
252

$\frac{35}{84} - 100$

$\frac{105}{252} - \frac{100}{252} = \frac{5}{252}$

35
3
105

<u>LMMSE</u>

$a_1 = \frac{\sigma_{xy}}{\sigma^2 x} = \frac{\frac{5}{252}}{\frac{5}{98}}$

$= \frac{98}{252} = \frac{49}{126} = \frac{7}{18}$

$a_0 = E[Y] - a_1 E[X]$

$= \frac{5}{6} - \frac{7}{18}\left(\frac{105}{243}\right)$

$= \frac{5}{6} - \frac{5}{27}$

$2 \cdot 3 \qquad 3^3$

$\frac{45}{54} - \frac{10}{54} = \frac{35}{54}$

$$\hat{X}_{LMMSE}(y) = a_0 + a_1 y$$

$$a_1 = \frac{\sigma_{xy}}{\sigma^2_y} \qquad a_0 = \mu_x - a_1 \mu_y$$

$$a_1 = \frac{\frac{5}{252}}{\frac{5}{252}} = 1.$$

$$a_0 = \frac{10}{21} - (1)\frac{5}{6}$$

$$= \frac{20}{42} - \frac{35}{42} = -\frac{15}{42}$$

$$= \frac{-5}{14}$$

$$MSE_{LMMSE} = E\left[\left(\hat{X}_{LMMSE}(y) - X\right)^2\right]$$

$$= \sigma^2_x (1 - \rho^2_{xy})$$

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{\frac{5}{252}}{\sqrt{\frac{5}{98}}\sqrt{\frac{5}{252}}} =$$

$$\frac{\sqrt{252 \cdot 98}}{252} = \sqrt{\frac{98}{252}} = \sqrt{\frac{7}{18}}$$

$$MSE_{LMMSE} = \frac{5}{98}\left(1 - \left(\sqrt{\frac{7}{18}}\right)^2\right)$$

$$= \frac{5}{98}\left(1 - \frac{7}{18}\right) = \frac{5}{98} \cdot \frac{11}{18}$$

$$= \frac{55}{1764}$$

$$\begin{array}{r} 6 \\ 98 \\ \times 18 \\ \hline 784 \\ 980 \\ \hline 1764 \end{array}$$