ECE302 – Quiz 2

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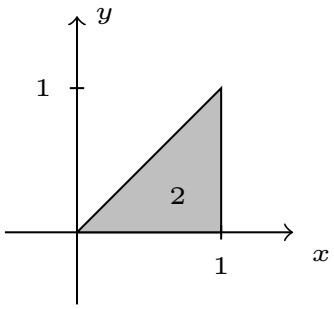
1. Consider random variables X and Y that have the following joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x < 1 \\ 0 & \text{else} \end{cases}$$

Drawing of support of X, Y:



In the following answers, the intervals for which the pdfs are well-defined are shown in parentheses following their equations; outside those intervals the pdfs have value 0.

(a) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y')\,dy' \\ &= \int_0^x 2\,dy' \\ &= 2y'\Big|_0^x = 2x \ (0 \le x < 1) \end{aligned}$$

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x', y)\, dx' \\
&= \int_{y}^{1} f_{X,Y}(x', y)\, dx' \\
&= \int_{y}^{1} 2\, dx' \\
&= 2x' \Big|_{y}^{1} = 2(1-y) \ (0 \le y < 1)
\end{aligned}
$$

(b) Find the conditional pdfs $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$.

$$
\begin{aligned}
f_{Y|X}(y \mid x) &= \frac{f_{X,Y}(x,y)}{f_x(x)} \\
&= \frac{2}{2x} = \frac{1}{x} \ (0 < y \le x \le 1) \\
f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_y(y)} \\
&= \frac{2}{2(1-y)} = \frac{1}{1-y} \ (0 \le y \le x < 1)
\end{aligned}
$$

(c) Find the expected values of the conditional pdfs $E[X \mid Y]$ and $E[Y \mid X]$.

$$
\begin{aligned}
E[X \mid Y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx \\
&= \int_{y}^{1} x \frac{1}{1-y}\, dx \\
&= \frac{1}{1-y} \ \frac{x^2}{2} \Big|_{x=y}^{1} = \frac{1^2 - y^2}{2(1-y)} \\
&= \frac{y+1}{2} \ (0 < y \le 1) \\
E[Y \mid X] &= \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)\, dy \\
&= \int_{0}^{x} y \frac{1}{x}\, dy \\
&= \frac{1}{x} \ \frac{y^2}{2} \Big|_{y=0}^{x} = \frac{x}{2} \ (0 < x \le 1)
\end{aligned}
$$