Stochastics Quiz #1
Name:
10 points total, 5 points each

Question 1: Suppose you have $63 and go to the casino, and play a game of roulette that only has 2 possible outcomes, red and green, such that P[red] = P[green] = ½. You employ the following strategy. First, you bet 1$ on red or green, if you guess correctly, you win $1, and leave the casino with $64. If you lose, you double down, and bet $2, in which case, if you guess correctly, you win $4. If you win, you take your winnings and go home. You keep doubling down until you either run out of money, or win once and then you stop.

a) What is the sample space of possible outcomes? (i.e. if you do this experiment, what possible winnings could you have?)
b) Write a PMF for the possible win amounts.
c) Find the average win amount?
d) Is this a good strategy? Should you play this game every day?

Question 2: *U* is a uniform random variable from (0,1). Let

$$X = -\ln(1-U)$$

Find the cdf and pdf of *X*

ECE302 – Quiz 1

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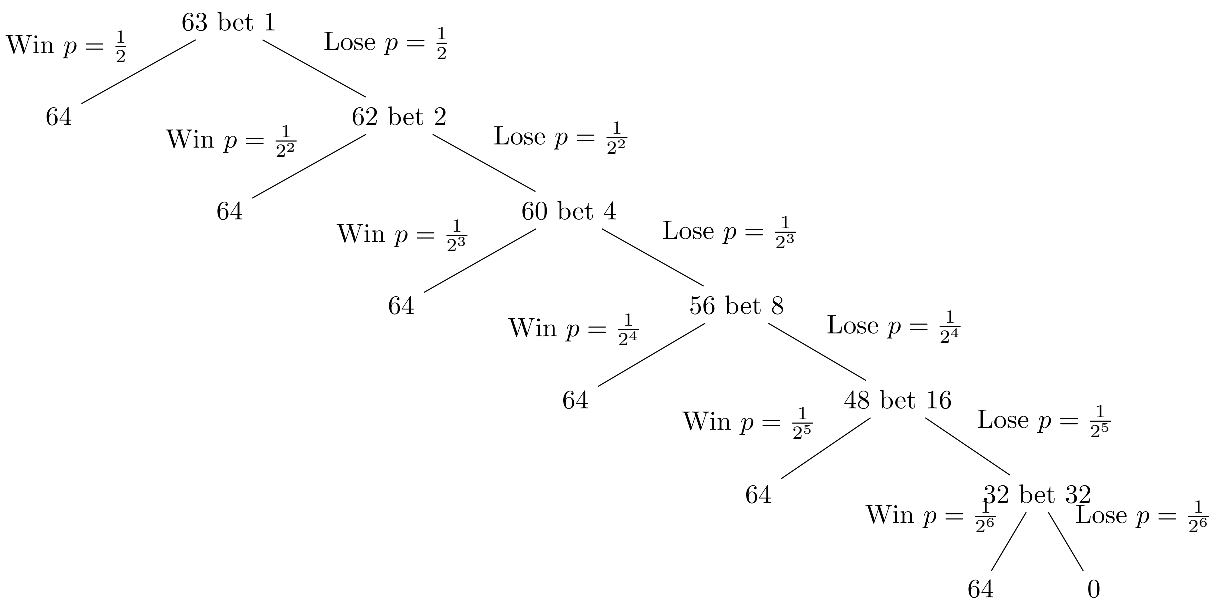
February 11, 2021

1. Your possible winnings look like:



   (a) There are two possible outcomes: win one dollar (leave with 64 dollars) or lose all 63 dollars. (I.e., it is essentially a Bernoulli trial.)

   (b) Looking at the leaf nodes, we can sum the probabilities of winning one dollar (all disjoint events) to get the total probability of winning one dollar, and there is only one event where we lose all 63 dollars. Let X denote the Bernoulli R.V. indicating our winnings.

$$\begin{aligned} P[X=1] &= \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6} &&= \frac{63}{64} \\ P[X=-63] &= \frac{1}{2^6} &&= \frac{1}{64} \end{aligned}$$

(c) Expected value of X:

$$E[X] = (1)\frac{63}{64} + (-63)\frac{1}{64} = \frac{63}{64} - \frac{63}{64} = 0$$

(d) Since the expected value of the winnings is breaking even, it is up to you to push your luck if you want to win money. (Personally I wouldn't bother with something that isn't expected to earn anything in the long run. There's a good chance that you'd hit the -63 dollars at some point and make it so that you can't even continue to play anymore unless you're willing to run some debt.)

2. If U is a uniform R.V. from $(0, 1)$, then its cdf is:

$$F_U(u) = P[U \leq u] = \begin{cases} 0 & u < 0 \\ u & 0 \leq u < 1 \\ 1 & u \geq 1 \end{cases}$$

The cdf of X is then:

$$\begin{aligned} F_X(x) &= P[X \leq x] \\ &= P[-\ln(1-U) \leq x] \\ &= P[\ln(1-U) \geq -x] \\ &= P[1 - U \geq e^{-x}] \\ &= P[U - 1 \leq -e^{-x}] \\ &= P[U \leq 1 - e^{-x}] \\ &= F_U(1 - e^{-x}) \\ &= \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases} \end{aligned}$$

(Checking bounds on X: X is only valid when $U < 1$, i.e., for $1 - e^{-x} < 1$, which is always true for all values of x.)

The pdf of X is the derivative of its cdf (use chain rule):

$$f_X(x) = \frac{dF_X}{dx} = \frac{dF_X}{d(1-e^{-x})}\frac{d(1-e^{-x})}{dx} = f_U(1-e^{-x})e^{-x}$$

Since $0 \leq 1 - e^{-x} < 1$ for positive x and $1 - e^{-x} < 0$ for negative x, and the pdf of U is:

$$f_U(u) = \begin{cases} 1 & 0 \leq u < 1 \\ 0 & \text{else} \end{cases}$$

then the pdf of X can be simplified to:

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$