ECE302 – Project 1 Analytical Results

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Accompanying code can be found on GitHub.

Let X_i be the random variable denoting the roll of a single die; let $Y_j = X_1 + X_2 + X_3$ be the random variable denoting the roll of three die to generate an ability score; let $Z = \max(Y_1, Y_2, Y_3)$ be the random variable denoting the maximum of three trials to generate an ability score (using the “fun” method). All X_i are independent uniform IID; thus all Y_j are independent uniform IID, and all Z_k are independent uniform IID.

1. (a) Three Bernoulli trials

$$P(Y = 18) = P(X_1 = 6, X_2 = 6, X_3 = 6) = \frac{1}{6}\frac{1}{6}\frac{1}{6} = \frac{1}{6^3}$$

(b) Complement of a binomial distribution

$$P(Z = 18) = 1 - P((Y_1 \neq 18) \wedge (Y_2 \neq 18) \wedge (Y_3 \neq 18)) = 1 - \left(1 - \frac{1}{6^3}\right)^3$$

(c) Each trait is a Bernoulli trial

$$P(Z_i = 18,\ 1 \leq i \leq 6) = (P(Z = 18))^6$$

(d) Get the set where all sums are ≤ 9, but at least one is equal to 9

$$\begin{aligned}
P(Z = 9) &= P(\text{all sums } \leq 9) - P((\text{all sums } \leq 9) \wedge (\text{no sums } = 9)) \\
&= P((Y_1 \leq 9) \wedge (Y_2 \leq 9) \wedge (Y_3 \leq 9)) \\
&- P((Y_1 \leq 8) \wedge (Y_2 \leq 8) \wedge (Y_3 \leq 8)) \\
&= \left(\frac{81}{216}\right)^3 - \left(\frac{56}{216}\right)^3
\end{aligned}$$

$$P(Z_i = 9,\ 1 \leq i \leq 6) = (P(Z = 9))^6$$

2. Let X_i denote the random variable representing the hitpoints (hp) of a goblin, and Y_j denote the random variable representing the damage (dmg) of a fireball shot. Note that $X_i \cup Y_j$ are mutually independent.

   (a) Normal expected value

$$\begin{aligned} E[X] &= \sum_x xP(X = x) \\ &= 1(0.25) + 2(0.25) + 3(0.25) + 4(0.25) \\ &= 2.5 \\ E[Y] &= \sum_y yP(Y = y) \\ &= 2(0.25) + 3(0.5) + 4(0.25) \\ &= 3 \\ E[Y > 3] &= P(Y = 4) = 0.25 \end{aligned}$$

   (b) We can enumerate the pmf by inspection

$$\begin{aligned} P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = 0.25 \\ P(Y = 2) = P(Y = 4) = 0.25 \\ P(Y = 3) = 0.5 \end{aligned}$$

   (c) We break up this question using a partition of Y.

$$\begin{aligned} P(\text{slay all 6}) &= P(Y \geq X_i \ \forall X_i) \\ &= P((Y \geq X \ \forall X_i | Y = 2) \vee (Y \geq X \ \forall X_i | Y = 3) \\ &\quad \vee (Y \geq X \ \forall X_i | Y = 4)) \\ &= \left(\frac{1}{2}\right)^6 (0.25) + \left(\frac{3}{4}\right)^6 (0.5) + 1^6 (0.25) \end{aligned}$$

(d) We can break down the event in question into a partition of three events (note that it is not possible that the surviving troll has ≤ 2 hp or that the firebolt did 4 dmg):

i. hp of surviving troll $= 4$, dmg $= 3$, all other trolls have hp ≤ 3
ii. hp of surviving troll $= 4$, dmg $= 2$, all other trolls have hp ≤ 2
iii. hp of surviving troll $= 3$, dmg $= 2$, all other trolls have hp ≤ 2

The probabilities of these events are easy to calculate.

Using Bayes' rule, we can calculate the posterior pmf of X given that five trolls didn't survive. Let W denote the event that the other five trolls died, and $W = \text{(i)} \cup \text{(ii)} \cup \text{(iii)} \Rightarrow P(W) = P(\text{(i)}) + P(\text{(ii)}) + P(\text{(iii)})$ (union becomes addition since the events are disjoint). Then:

$$P(X = x|W) = \frac{P((X = x) \wedge W)}{P(W)}$$

This in turn can be used to calculate the expected hp of the surviving troll:

$$P(\text{(iii)}) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^5$$
$$P(\text{(ii)}) = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)^5$$
$$P(\text{(i)}) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^5$$
$$P(X = 3|W) = \frac{P(\text{(iii)})}{P(W)}$$
$$P(X = 4|W) = \frac{P(\text{(i)} \cup \text{(ii)})}{P(W)}$$
$$E[X|W] = \frac{1}{P(W)}\left[(3)P(X = 3|W) + (4)P(X = 4|W)\right]$$

(e) Let Z_i denote the random variable denoting a roll of the 20-sided die (to decide whether Shedjam can hit Keene or not), W_j denote a roll of the 6-sided die (the Sword of Tuition's damage), and V_k denote a roll of the 4-sided die (the Hammer of Tenure Denial's damage).

$$\begin{aligned}
E[\text{dmg}] &= E[\text{dmg}_{SoT} + \text{dmg}_{HTD}] \\
&= P(\text{hit}_{SoT})E[\text{dmg}_{SoT}|\text{hit}_{SoT}] + P(\text{hit}_{HTD})E[\text{dmg}_{HTD}|\text{hit}_{HTD}] \\
&= \left(\frac{10}{20}\right)(3.5 + 3.5) + \left(\frac{10}{20}\frac{10}{20}\right)(2.5)
\end{aligned}$$