ECE302 – HW3

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\begin{aligned}
f_{X,Y}(x,y) &= 2 \\
f_X(x) &= 2x \\
f_Y(y) &= 2(1-y) \\
f_{X|Y}(x \mid y) &= \frac{1}{1-y} \\
f_{Y|X}(x \mid y) &= \frac{1}{x} \\
\mu_X &= E[X] = \frac{2}{3} \\
\mu_Y &= E[y] = \frac{1}{3} \\
E[X^2] &= \frac{1}{2} \\
E[Y^2] &= \frac{1}{6} \\
\sigma_X^2 &= E[X^2] - \mu_X^2 = \frac{1}{18} \\
\sigma_Y^2 &= E[Y^2] - \mu_Y^2 = \frac{1}{18} \\
E[XY] &= \frac{1}{4} \\
\sigma_{XY}^2 &= E[XY] - \mu_X \mu_Y = \frac{1}{36} \\
\rho_{XY} &= \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y} = \frac{1}{2}
\end{aligned}
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1. $f_X(x) = 2x$, $f_{Y|X}\left(y \mid \frac{1}{3}\right) = \frac{1}{\frac{1}{3}} = 3$
2. No. $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$
3. $$\hat{Y}_{\text{MMSE}}(X) = E[Y \mid X] = \int_0^x y f(y \mid x)\, dy = \frac{X}{2}$$

4.

$$\text{MSE} = E[(\hat{Y}_{\text{MMSE}}(X) - Y)^2] = E\left[\left(\frac{X}{2} - Y\right)^2\right] = \frac{1}{24}$$

$$b = E[\hat{Y} - Y] = 0 \Rightarrow \text{ unbiased (as expected)}$$

5.

$$\begin{aligned}\hat{Y}_{\text{LMMSE}}(X) &= \mu_Y + \rho_{XY}\frac{\sigma_Y}{\sigma_X}(X - \mu_X) \\ &= \frac{X}{2} \quad (= \hat{Y}_{\text{MMSE}}(X)) \\ \text{MSE} &= \sigma_Y^2(1 - \rho_{XY}^2) = \frac{1}{24} \text{ (same as before)}\end{aligned}$$