ECE300 – Pset 4

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October 27, 2020

1. *I have a baseband analog signal with a* 20kHz *bandwidth. How fast must I sample to ensure a* 4kHz *guard band?*

   A 4kHz guard band means that the Nyquist bandwidth should be at $(20+4)\text{kHz} = 24\text{kHz}$, and the Nyquist rate should be $2 \times 24\text{kHz} = 48\text{kHz}$.

2. *Suppose I have bandlimited noise with PSD* 8 *for frequency less than* 200kHz *in absolute value. Sampling this noise at Nyquist and applying a* 16*-level quantizer, what are the rate and distortion (mean squared error)? What is the SQNR?*

   The Nyquist rate is $2 \times 200\text{kHz} = 400\text{kHz}$. Sampling at this rate, with 16 levels (4 bits) per sample, yields a bit transmission rate of $4\text{b} \times 400\text{kHz} = 1.6\text{Mbps}$.

   Denote the random process as X, and the quantization function Q. We're given the PSD of X:

   $$S_X(f) = \begin{cases} 8, & |f| < 200000 \\ 0, & \text{else} \end{cases}$$

   The mean squared error D is defined as follows:

   $$D = E[(X - Q(X))^2] = \int_{-\infty}^{\infty} (x - Q(x))^2 f_X(x)\, dx$$

   Now we need to find the PDF f_X. Assuming that the noise is white and gaussian, and knowing that the autocorrelation function of white gaussian noise is a delta with height σ_X^2 at $\tau = 0$, and knowing that the autocorrelation function is the inverse Fourier transform of the PSD:

   $$\begin{aligned} \sigma^2 = R_X(\tau)\Big|_{\tau=0} &= \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau}\, df \Big|_{\tau=0} \\ &= \int_{-200000}^{200000} (8)(1)\, df = 3200000 \end{aligned}$$

   Plugging this into the PDF of a gaussian:

   $$f_X(x) = \frac{1}{\sqrt{2\pi \times 3200000}} \exp \frac{x^2}{2 \times 3200000}$$

Finally, substituting this into the mean squared error formula:

$$\begin{aligned} D &= \int_{-\infty}^{\infty} (x - Q(x))\, f_X(x)\, dx \\ &= \sum_{i=1}^{16} \int_{R_i} (x - \hat{x}_i)^2\, f_X(x)\, dx \end{aligned}$$

where R_i and x_i denote the quantization intervals and levels, respectively. Without knowing the quantization function, this is as far as we know.

Plugging into the definition for SQNR:

$$\text{SQNR} = \frac{E[X^2]}{E[(X - Q(X))^2]} = \frac{R_X(0)}{D} = \frac{3200000}{\sum_{i=1}^{16} \int_{R_i} (x - \hat{x}_i)^2\, f_X(x)\, dx}$$

The next few problems will consider the following setup: Suppose I am doing a digital logic design project, and I represent 1 *as a* 2.5V *pulse of duration* A *and* 0 *as a* -2.5V *pulse of duration* A*. My chips perform a least squares decision when they receive a signal. The noise over a wire is largely thermal noise caused by statistical variations in charge carriers, modeled as AWGN. This noise has variance* $4k_BTR$ *where* $k_B \approx 1.38 \times 10^{-23} J/K$ *is the Boltzmann constant,* T *is the absolute temperature (in Kelvin) and* R *is the total resistance of the wire. A commonly used wire type (available in the lab at school) is* 22 *AWG solid copper wire, for which* 1m *of wire has* 52.7mΩ *resistance. Recall that resistance is proportional to length.*

3. *Describe the operation of a least squares decision system for this problem.*

   If you had some way to measure the average voltage of the signal over the period $t \in [0, A]$, and see if that value is positive or negative (where positive indicates the positive pulse, and negative indicates the negative pulse), this would perform the least squares decision system. (This is closer to doing a matched filter (convolution with a rectangular pulse should give some measure of the average value over that pulse), but as stated below in (4), this is equivalent to a least-squares decision boundary. Clearly, the decision boundary is at zero average voltage.)

4. *Is the least squares decision rule the same as a matched filter decision rule in this case? Is it the same as maximum likelihood? Is it the same as maximum a posteriori? If there is not enough information to answer, give the conditions that would guarantee the equality.*

   It is equivalent to MF, since the two pulses are equal-energy. The inner product is a measure of distance if the units are properly normalized.

   It is equivalent to ML, since the PDF of the noise is symmetric (since it is gaussian). This is clear from a geometric perspective since the halfway point is in the middle due to symmetry.

It is equivalent to MAP iff both the positive and negative pulses are equiprobable. ML is MAP with the assumption that the priors are equal (events are equiprobable).

5. *What is the name of this modulation scheme? What is the basis? Draw the geometrical representation.*

   This is the binary antipodal signaling scheme (a form of PAM).

   The (single) basis vector is the positive signal (a rectangular pulse of height 2.5 and width A, denote this S_1) normalized to unit energy. (The choice of positive or negative symbol is arbitrary.) I.e.:

$$\psi(t) = \frac{S_1(t)}{||S_1(t)||} = \frac{S_1(t)}{\sqrt{\int_0^A 2.5^2 \, dt}} = \frac{1}{2.5\sqrt{A}} S_1(t)$$

   Geometrical representation: along a 1-D line representing the ψ axis, the two signals are located at $\pm||S_1(t)|| = \pm\sqrt{\varepsilon_1}$.

6. *For this scheme, suppose I transmit a* 1 75% *of the time – what are the MAP decision regions?*

   MAP minimizes the probability of error, so the decision boundary should lie at the point where the probability of the transmitted signal being either 0 or 1 to be equal. Let y denote the signal in the ψ representation, and $\hat{x}$ denote the transmitted signal. Let α denote the decision boundary.

$$\begin{aligned}
P_{err} &= P[y > 0|\hat{x} = 0]P[\hat{x} = 0] + P[y < 0|\hat{x} = 1]P[\hat{x} = 0] \\
&= P\left[N\left(-2.5\sqrt{A}, 2\sqrt{k_B T R}\right) > 0\right](0.25) + P\left[N\left(2.5\sqrt{A}, 2\sqrt{k_B T R} < 0\right)\right](0.75) \\
&= \frac{1}{4}\frac{1}{2\pi(4k_B T R)}\int_{-\infty}^{\alpha} \exp\frac{-(x+2\sqrt{A})^2}{2(4k_B T R)}\,dx + \frac{3}{4}\frac{1}{2\pi(4k_B T R)}\int_{\alpha}^{\infty} \exp\frac{-(x-2\sqrt{A})^2}{2(4k_B T R)}\,dx
\end{aligned}$$

   This becomes an optimization problem to find $\operatorname{argmin}_\alpha P_{err}$.

$$\begin{aligned}
0 &= \frac{\partial P_{err}}{\partial \alpha} \\
&= \frac{1}{\sqrt{2\pi(4k_B T R)}}\left[\frac{1}{4}\exp\frac{-(\alpha+2.5\sqrt{A})^2}{2(4k_B T R)} - \frac{3}{4}\exp\frac{-(\alpha-2.5\sqrt{A})^2}{2(4k_B T R)}\right] \\
\frac{3}{1} &= \frac{\exp\frac{-(\alpha+2.5\sqrt{A})^2}{2(4k_B T R)}}{\exp\frac{-(\alpha-2.5\sqrt{A})^2}{2(4k_B T R)}} = \exp\frac{-4\alpha(2.5\sqrt{A})}{2(4k_B T R)} \\
\alpha &= -\frac{4k_B T R}{5\sqrt{A}}\log(3)
\end{aligned}$$

   It makes sense that this value is negative, as the probability of transmitting a 1 is more likely.

7. *Assume now that the symbols are equiprobable. In terms of A, T and wire length L, write a formula for the SNR and probability of error.*

   Since resistance is proportional to length, define the resistivity ρ to be the constant of proportion:

$$\begin{aligned} R &\propto L \\ R &= \rho L \\ \rho &= \frac{R}{L} = \frac{52.7\text{m}\Omega}{1\text{m}} = 52.7 \times 10^{-3}\Omega\text{m}^{-1} \end{aligned}$$

   SNR calculation (use the MF version here, since it is equivalent):

$$\text{SNR} = \frac{\varepsilon_y}{\frac{N_0}{2}} = \frac{2.5^2 A}{4k_B TR}$$

   Probability of error calculation:

$$\begin{aligned} P_{err} &= P[y > 0|\hat{x} = 0]P[\hat{x} = 0] + P[y < 0|\hat{x} = 1]P[\hat{x} = 0] \\ &= P[y > 0|\hat{x} = 0] \\ y|(x = 0) &\sim N\left(-2.5\sqrt{A}, 2\sqrt{k_B TR}\right) \\ P_{err} &= Q\left(\frac{0 - \left(-2.5\sqrt{A}\right)}{2\sqrt{k_B TR}}\right) \\ &= Q\left(\frac{5}{4}\sqrt{\frac{A}{k_B TR}}\right) \\ &= Q\left(\frac{5}{4}\sqrt{\frac{A}{k_B T\rho L}}\right) \end{aligned}$$

8. *Assume we are operating the system at room temperature – how long does the wire have to be to yield an error probability of* $1/10$*? Write this number in lightyears. Compare this to the size of the Milky Way. This should give you an appreciation for how robust wired communication is to thermal noise*

   A value for A was never given, so arbitrarily let $A = 1$s. Also, let the room temperature be 25°C. Plugging into the error formula:

$$
\begin{aligned}
\frac{1}{10} &= Q\left(\frac{5}{4}\sqrt{\frac{A}{k_B T \rho L}}\right) \\
L &= \left(\frac{4}{5}\sqrt{\frac{k_B T \rho}{A}} Q^{-1}\left(\frac{1}{10}\right)\right)^{-2} \\
&= \left(\frac{4}{5}\sqrt{\frac{(1.38\times 10^{-23}\mathrm{J/K})(298K)(52.7\times 10^{-3}\Omega\mathrm{m}^{-1})}{(1\mathrm{s})}} Q^{-1}\left(\frac{1}{10}\right)\right)^{-2} \\
&= 2.66\times 10^{21}\mathrm{m} = 4.39\times 10^{5}\mathrm{ly}
\end{aligned}
$$

That is long. The Milky Way's diameter, in comparison, is roughly 1.06×10^5ly, so they are on the same order of magnitude.

Granted, a one second pulse may be considered slow by today's standards. By the above formula, it is not hard to see that $L \propto A$. Even if our device was clocked in the microsecond range $A \approx 1\mu$s, the length would be 4.39×10^{-1}ly, still formidable.