ECE300 – Pset 3

Jonathan Lam

September 26, 2020

1. *A source generates 0s and 1s randomly according to a Bernoulli distribution, where the probability of a 0 is 0.3 and the probability of a 1 is 0.7. The value is then sent across a long wire, and corrupted by thermal noise. A system at the other end receives the corrupted signal and guesses if it received a 1 or a 0. It has an error probability (either guessing a 1 was a 0 or guessing a 0 was a 1) of 0.2. If the receiver guesses it received a 1, what is the probability a 1 was transmitted?*

   Let T represent the transmitted (true) value, and G represent the received (guessed) value.

$$\begin{aligned}\Pr(T=0)&=0.3\\ \Pr(T=1)&=0.7\\ \Pr(G=1|T=0)=\Pr(G=0|T=1)&=0.2\end{aligned}$$

   Use Bayes' Theorem (where $T=\{0,1\}$ is the partition):

$$\begin{aligned}\Pr(T=1|G=1)&=\frac{\Pr(T=1\cap G=1)}{\Pr(G=1)}=\frac{\Pr(G=1|T=1)\Pr(T=1)}{\sum_{i=0}^{1}\Pr(G=1|T=i)\Pr(T=i)}\\ &=\frac{(1-0.2)(0.7)}{(0.2)(0.3)+(1-0.2)(0.7)}=\frac{28}{31}\end{aligned}$$

2. *Let θ be a random variable uniformly distributed from 0 to π. Let $X = \cos\theta$ and $Y = \sin\theta$. Are X and Y uncorrelated? Are they independent? Are they orthogonal?*

$$
\begin{aligned}
f_\theta(\tau) &= \frac{1}{\pi} && (\tau \in [0, \pi]) \\
\mu_X = E[X] &= \int_0^\pi X(\theta = \tau) f_\theta(\tau)\, d\tau = \frac{1}{\pi}\int_0^\pi \cos\tau\, d\tau = 0 \\
\mu_Y = E[Y] &= \int_0^\pi Y(\theta = \tau) f_\theta(\tau)\, d\tau = \frac{1}{\pi}\int_0^\pi \sin\tau\, d\tau = \frac{2}{\pi} \\
r_{X,Y} = E[XY] &= \int_0^\pi X(\theta = \tau) Y(\theta = \tau) f_\theta(\tau)\, d\tau \\
&= \frac{1}{\pi}\int_0^\pi \cos\tau \sin\tau\, d\tau = 0 && (1) \\
\operatorname{cov}(X, Y) &= E[XY] - \mu_X \mu_Y = 0 - (0)\left(\frac{2}{\pi}\right) = 0 && (2) \\
f_X(x) &= \frac{d}{dx} F_X(x) = \frac{d}{dx}\Pr(\cos\theta < x) = \frac{d}{dx}\Pr(\theta > \operatorname{acos} x) = \\
&= \frac{d}{dx}\frac{1}{\pi}\int_{\operatorname{acos} x}^{\pi} d\tau = \frac{1}{\pi}\frac{d}{dx}[\pi - \operatorname{acos} x] = \frac{1}{\pi\sqrt{1 - x^2}} && (x \in [-1, 1]) \\
f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy}\Pr(\sin\theta < y) = 2\frac{d}{dy}\Pr(\theta < \operatorname{asin} y) = \\
&= \frac{d}{dy}\frac{2}{\pi}\int_0^{\operatorname{asin} y} d\tau = \frac{d}{dy}\frac{2}{\pi}[\operatorname{asin} y] = \frac{2}{\pi\sqrt{1 - y^2}} && (y \in [0, 1]) \\
f_{X,Y}(x, y) &= \begin{cases} 1 & x^2 + y^2 = 1 \\ 0 & else \end{cases} && (x \in [-1, 1], y \in [0, 1]) \\
f_X(x) f_Y(y) &\neq f_{X,Y}(x, y) && (3)
\end{aligned}
$$

Uncorrelated (2), dependent (3), orthogonal (1). Orthogonality makes sense because sin and cos are orthogonal functions, dependence makes sense because X and Y rely on the same random variable (i.e., if you set X, then you fix the value of θ and thus also the value of Y), uncorrelatedness makes sense because an increase in X doesn't tend to cause an increase in Y.

3. *Let X(t) be a random process defined by $X(t) = A + Bt$ where A and B are independent random variables uniformly distributed from -1 to 1. Find $m_X(t)$ and $R_X(t_1, t_2)$. Is the process WSS? If not, is it cyclostationary? If the answer to either question is yes, find the PSD of X.*

$$
\begin{aligned}
f_A(c) &= f_B(c) = \frac{1}{2} && (c \in [-1, 1]) \\
\mu_A &= E[A] = \frac{1}{2}\int_{-1}^{1} a\, da = 0 = \mu_B \\
m_X(t) &= E[A + Bt] = E[A] + tE[B] = \mu_A + t\mu_B = 0 && (4) \\
E[A^2] &= \frac{1}{2}\int_{-1}^{1} a^2\, da = \frac{1}{3} = E[B^2] \\
E[AB] &= E[A]E[B] = 0 && \text{(by independence)} \\
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] = E[(A + Bt_1)(A + Bt_2)] && (5) \\
&= E[A^2 + AB(t_1 + t_2) + Bt_1t_2] \\
&= E[A^2] + (t_1 + t_2)E[AB] + t_1t_2E[B^2] \\
&= \frac{1}{3} + (0)(t_1 + t_2) + \frac{1}{3}t_1t_2 = \frac{1}{3}(1 + t_1t_2) \\
R'_X(t) &= R_X(t + \tau, t) = \frac{1}{3}(1 + t^2 + \tau t) && (6)
\end{aligned}
$$

The mean is not a function of time (4) but the autocorrelation is not a function of the delay $t_1 - t_2$ (5), so X is not WSS. If we express the autocorrelation as a function of the starting time $R'_X(t)$ (where $\tau = t_2 - t_1$ is the delay), we see that it is not periodic for any period, so it is not cyclostationary.

4. *Let $X(t) = Y\cos(\omega_0 t) - Z\sin(\omega_0 t)$, where Y and Z are zero-mean independent gaussians with variance σ^2. Find $m_X(t)$ and $R_X(t_1, t_2)$. Is the process WSS? If not, is it cyclostationary? If the answer to either question is yes, find the PSD of X.*

$$
\begin{aligned}
0 &= E[Y] = E[Z] \\
\sigma^2 &= E[(Y - \mu_Y)^2] = E[(Z - \mu_Z)^2] = E[Y^2] = E[Z^2] \\
m_X(t) &= E[X] = E[Y]\cos\omega_0 t + E[Z]\omega_0 t = 0 && (7) \\
E[YZ] &= E[Y]E[Z] = 0 && \text{(by independence)} \\
R_X(t_1, t_2) &= E[(Y\cos\omega_0 t_1 + Z\sin\omega_0 t_1)(Y\cos\omega_0 t_2 + Z\sin\omega_0 t_2)] && (8) \\
&= \cos(\omega_0 t_1)\cos(\omega_0 t_2)E[Y^2] + E[YZ]\cos(\omega_0 t_1)\sin(\omega_0 t_2) \\
&\quad + E[YZ]\sin(\omega_0 t_1)\cos(\omega_0 t_2) + E[Z^2]\sin(\omega_0 t_1)\sin(\omega_0 t_2) \\
&= \sigma^2(\cos(\omega_0 t_1)\cos(\omega_0 t_2) + \sin(\omega_0 t_1)\sin(\omega_0 t_2)) + (0)(\dots) \\
&= \sigma^2\cos(\omega_0(t_1 - t_2)) = R_X(t_2 - t_1) = R_X(\tau)
\end{aligned}
$$

The mean is constant (7) and the autocorrelation is a function of time lag τ only (8), so it is WSS (and thus also cyclostationary).

$$
S_X(\omega) = \mathcal{F}\{R_X(\tau)\}(\omega) = \frac{\sigma^2\pi}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \qquad \text{(W-K Thm.)}
$$

5. *If $X(t)$ has PSD $S_X(\omega)$, what is the PSD of $Y(t) = 4X'(t) + X(t-T)$? (Assume X is WSS.)*

$$\begin{aligned}
S_Y(t) &= E[|Y(\omega)|^2] \\
&= E\left[\left|4j\omega X(\omega) + X(\omega)e^{-j\omega T}\right|^2\right] \\
&= E\left[|X(\omega)|^2 \left|4j\omega + e^{-j\omega T}\right|^2\right] \\
&= \left|4j\omega + e^{j\omega T}\right|^2 E\left[|X(\omega)|^2\right] \\
&= \left(4j\omega + e^{-j\omega T}\right)\left(-4j\omega + e^{j\omega T}\right) S_X(t) \\
&= \left(16\omega^2 - 8\omega\left(\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right) + 1\right) S_X(t) \\
&= \left(16\omega^2 - 8\omega \sin \omega T + 1\right) S_X(t)
\end{aligned}$$

6. *In the discrete-time case, suppose we have a random process defined by* $\{X_n\}$ *for* $n \in \mathbb{Z}$*. If we observe* N *samples of the random process,* $\{x_n\}$ *for* $n = 1, 2, \ldots, N$*, we define the* sample autocorrelation *as*

$$R_X(m) = \begin{cases} \frac{1}{N-m} \sum_{n=1}^{N-m} x_n x_{n+m}, & m = 0, 1, \ldots \\ \frac{1}{N-|m|} \sum_{n=|m|+1}^{N} x_n x_{n+m}. & m - 1, -2, \ldots \end{cases}$$

This approximates the autocorrelation. In practice, we don't let m take infinitely many values, but instead look at it over a finite range of values from -M to M for some integer M . The Power Spectral Density is computed through the Wiener Khinchin Theorem, using the DFT rather than the CTFT:

$$S_X(\omega) = \sum_{m=-M}^{M} R_X(m) \exp \frac{-j\omega m}{2M+1}$$

Using MATLAB, write a function that takes an input vector of N samples and an integer M as inputs, and returns the autocorrelation and PSD.

```
% X: 1xN
% m: scalar
% returns: scalar
function res = autocorr(X, m)
    m = abs(m);
    N = length(X);
    res = X(1:N-m) * X(m+1:N).' / (N - m);
end

% X: 1xN
% M: scalar
% w: 1xW
% returns: 1xW
function res = psd(X, M, w)
    m = -M:M;
    res = arrayfun(@(m) autocorr(X, m), m) * exp(-1j * w .* m.' / (2 * M + 1));
end
```

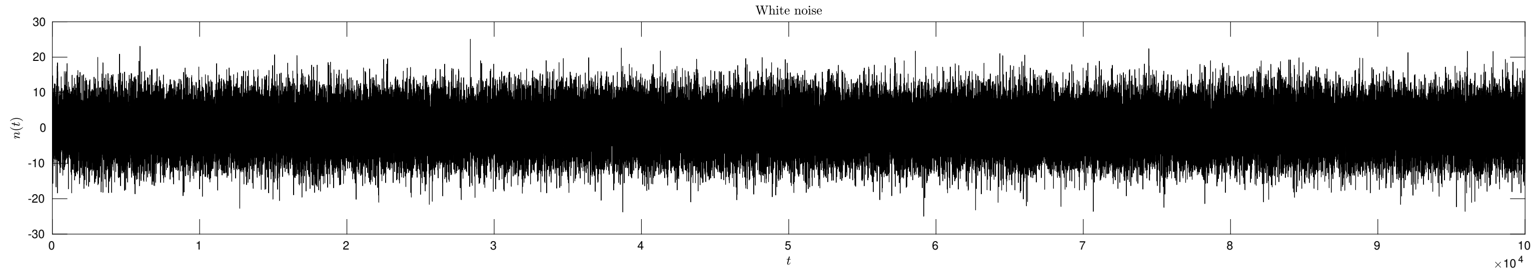
Notes on implementation:

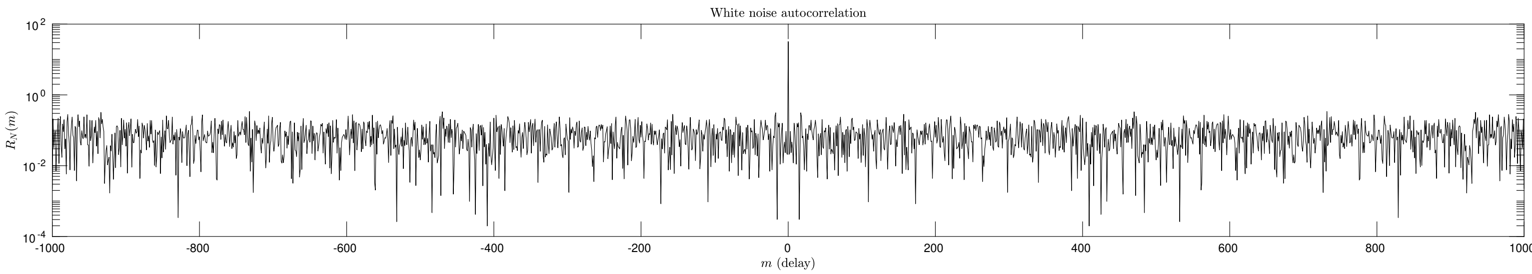
- Since R_X seems to be symmetric about $m = 0$, I wrote it without breaking up the cases.
- This `autocorr` implementation takes a signal and a scalar m, but the `psd` function can take an array of ω values over which it estimates the PSD. The reason for this difference is that while we can neatly broadcast the multiplication in the PSD function, but the slices of X in `autocorr` are different lengths and cannot be broadcasted (as far as I know); thus they are mapped over with an `arrayfun` operation.

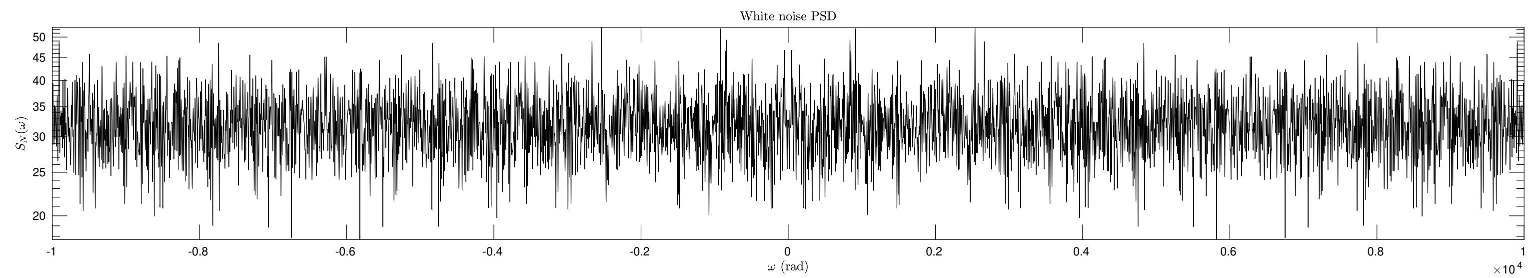
7. *Use your function from above to plot the power spectral density white noise with some variance you choose so as to validate your function's operation. Make sure to use sufficiently large N and M.*

```
N = 100000;                         % number of samples
M = 1000;                           % autocorrelation range
n_variance = 32;                    % variance of white noise
w = -10000:10000;                   % frequency axis for PSD

% generate white noise, plot it
n_sig = randn(1, N) * sqrt(n_variance);
subplot(3, 1, 1);
plot(n_sig);
title('White noise');
ylabel('$n(t)$');
xlabel('$t$');

% compute and plot autocorrelation of white noise
m = -M:M;
R_N = arrayfun(@(m) autocorr(n_sig, m), m);
subplot(3, 1, 2);
semilogy(m, abs(R_N));
title('White noise autocorrelation');
ylabel('$R_N(m)$');
xlabel('$m$ (delay)');

% compute and plot PSD of white noise, plot it
S_N = psd(n_sig, M, w);
subplot(3, 1, 3);
semilogy(w, abs(S_N));
title('White noise PSD');
ylabel('$S_N(\omega)$');
xlabel('$\omega$ (rad)');
```

(See figure on next page.)





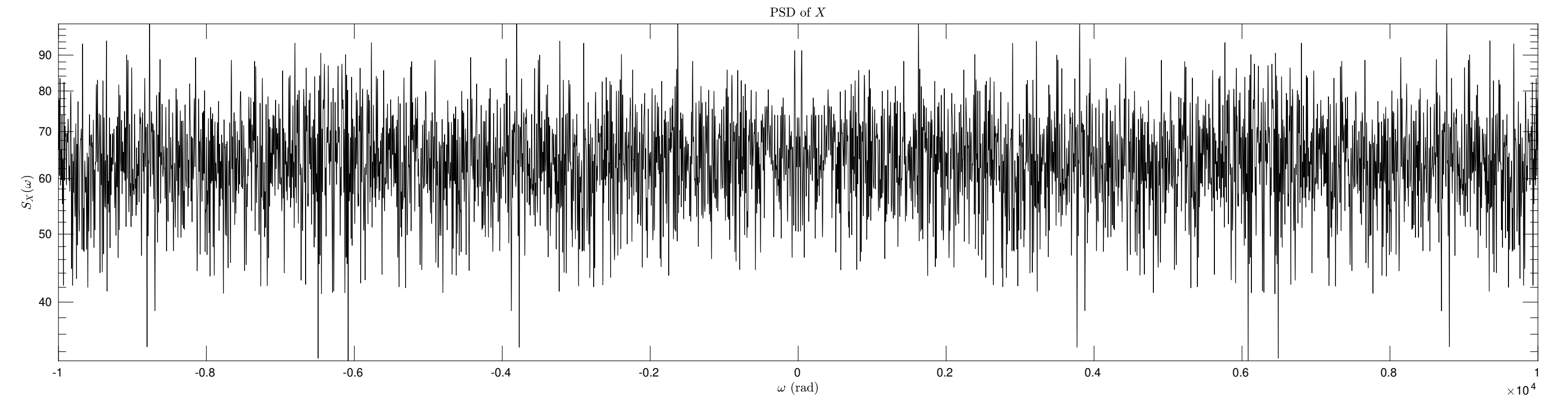


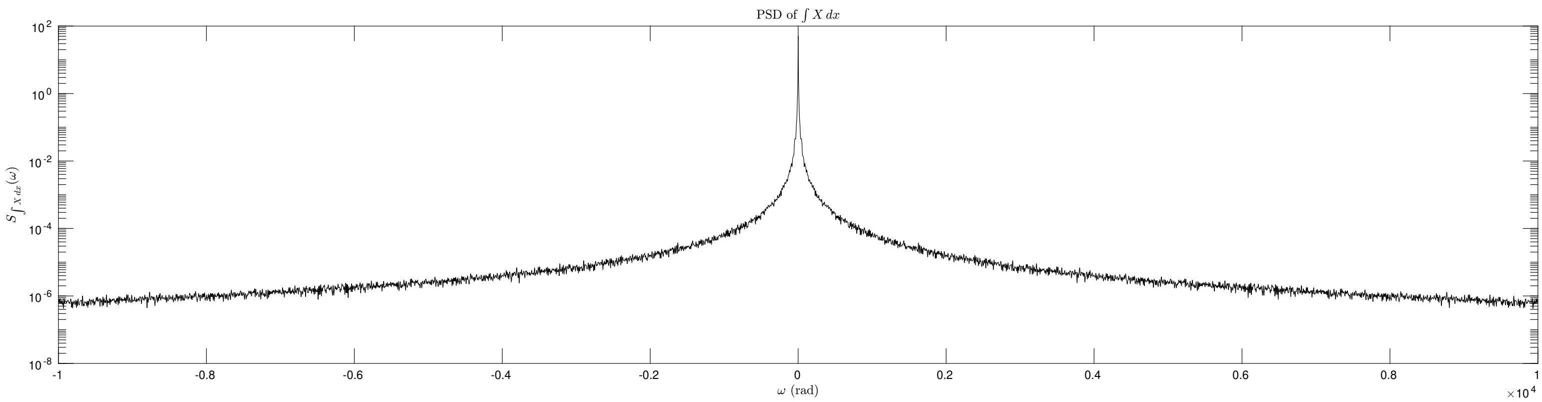
- The white noise looks about correct.
- The autocorrelation of the white noise is mostly ≈ 0 (the plot axes are semilogy), with a sharp peak at $\omega = 0$, which jives with our understanding of white noise. The peak's magnitude is roughly that of the chosen variance σ^2 (arbitrarily chosen to be 32).
- The PSD is roughly centered about the variance and shows no pattern. While it looks very jagged here because the y-axis scale is narrow, it is arguably fairly "flat."

8. *Use your function to plot the PSD of the of $X(t)$ from problem 4 with ω =10000 rad/s, as well as the PSD of the integral of $X(t)$. Please don't perform an integral numerically (except to check your work if you want to) – instead, use the relationship of PSDs at the input and output of an LTI system.*

```
w_0 = 10000;                            % cosine frequency (Q8)
YZ_var = 64;                            % variance for Y and Z RVs (Q8)

% X(t) = Y*cos(w_0*t) - Z*sin(w_0*t), w_0 = 10000rad/s
t = 0:1/N:1;
Y = randn(size(t)) * sqrt(YZ_var);
Z = randn(size(t)) * sqrt(YZ_var);
X = Y .* cos(w_0 * t) - Z .* sin(w_0 * t);

% compute and plot PSD of X
S_X = psd(X, M, w);
figure();
subplot(2, 1, 1);
semilogy(w, abs(S_X));
title('PSD of $X$');
ylabel('$S_X(\omega)$');
xlabel('$\omega$ (rad)');

% compute and plot PSD of integral of X
S_Y = S_X ./ (w .^ 2);
subplot(2, 1, 2);
semilogy(w, abs(S_Y));
title('PSD of $\int X\,dx$');
ylabel('$S_{\int X\,dx}(\omega)$');
xlabel('$\omega$ (rad)');
```

(See figure on next page.)





- Like in the previous question, the PSD is relatively flat and centered around the variance (arbitrarily chosen to be 64 this time).
- The PSD of the integral of X was generated using the rule based on the Wiener-Khinchin Theorem, namely:

  $$S_Y = |H|^2 S_X$$

  In this case, the LTI system H has the magnitude response

  $$|H|^2 = \left|\frac{1}{j\omega}\right|^2 = \frac{1}{\omega^2}$$

  which, when multiplied with the relatively-flat PSD of X, gives the PSD in the second plot.