ECE300 - Pset 1

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1. *A square wave has period T, amplitude A and duty cycle τ/T (the signal takes value A from time 0 to time τ, then 0 from time τ to T). Find the Fourier series representation of this signal.*

   Let the square wave be $x(t)$, and let c_n denote the Fourier coefficients.

$$\begin{aligned}
c_n &= \frac{1}{T}\int_0^T x(t)\exp(-j\omega nt)\,dt \\
&= \frac{1}{T}\left[\int_0^\tau A\exp(-j2\pi nt/T)\,dt + \int_\tau^T (0)\exp(-j2\pi nt/T)\,dt\right] \\
&= \frac{1}{T}\left[A\int_0^\tau \exp(-j2\pi nt/T)\,dt + 0\right] \\
&= j\frac{A}{T}\frac{T}{2\pi n}\exp(-j2\pi nt/T)\Big|_{t=0}^{t=\tau} \\
&= j\frac{A}{2\pi n}(\exp(-j2\pi n\tau/T) - 1)
\end{aligned}$$

   Note that this general formula doesn't work for the $n = 0$ case, so it needs to be done manually:

$$c_0 = \frac{1}{T}\int_0^\tau A\exp(0)dt = \frac{A\tau}{T}$$

   Plugging into the Fourier series:

$$\begin{aligned}
x(t) &= \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi nt/T} \\
&= c_0 e^0 + \frac{A}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n} e^{j\pi/2}(e^{-j2\pi n\tau/T} - e^0)e^{j2\pi nt/T} \\
&= \frac{A\tau}{T} + \frac{A}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n}\left(e^{j(\pi/2+2\pi n/T(t-\tau))} - e^{j(\pi/2+2\pi nt/T)}\right)
\end{aligned}$$

Alternatively, to express this as a real signal, we know that $c_n = c_{-n}^*$, and therefore the summand $s_n := c_n e^{j2\pi nt/T}$ also has this conjugate symmetry:

$$s_n = c_n e^{j\pi nt/T} = \left((c_n^*) \left(e^{j2\pi nt/T} \right)^* \right)^* = \left(c_{-n} e^{j2\pi(-n)t/T} \right)^* = s_{-n}^*$$

Since $s_n + s_{-n} = s_n + s_n^* = 2 \operatorname{Re} s_n$, the value of the complex sum is the same as twice the real part of the summand for the nonzero terms. Thus, the Fourier series in terms of real functions only is:

$$x(t) = \frac{A\tau}{T} + (2)\frac{A}{2\pi} \sum_{n=1}^{+\infty} \frac{1}{n} \left(\cos\left(\frac{\pi}{2} + \frac{2\pi n}{T}(t-\tau) \right) - \cos\left(\frac{\pi}{2} + \frac{2\pi n}{T} t \right) \right)$$

2. *Does the signal from the previous question have finite energy? Does it have finite power? If the answer to either question is yes, find the value of the energy or power.*

   The signal has infinite energy (since it is periodic, and thus has a nondiminishing average value even as $t \to \pm\infty$). It does have finite power. Since it is periodic, we only have to calculate the power over one period:

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt = \frac{1}{T} \left[\int_0^\tau A^2 dt + \int_\tau^T 0^2 dt \right] = \frac{A^2 \tau}{T}$$

3. *What is* $\operatorname{sinc}(t) * \operatorname{sinc}(t)$*?*

   The Fourier transform of a (normalized) sinc function is the rect function. The (pointwise) product of the rect function with itself is itself, and the inverse Fourier transform of itself is the normalized sinc function. So, in this case the convolution of sinc with itself is itself.

$$\begin{aligned}
\operatorname{sinc}(t) &:= \frac{\sin \pi t}{\pi t} \\
\operatorname{rect}(f) &:= \begin{cases} 1 & -1/2 \le f < 1/2 \\ 0 & \text{else} \end{cases} \\
\mathcal{F}\{\operatorname{sinc}(t)\}(f) &= \operatorname{rect}(\omega) \Leftrightarrow \mathcal{F}^{-1}\{\operatorname{rect}(f)\}(t) = \operatorname{sinc}(t) \\
\operatorname{sinc}(t) * \operatorname{sinc}(t) &= \mathcal{F}^{-1}\{\mathcal{F}\{\operatorname{sinc}(t) * \operatorname{sinc}(t)\}(f)\}(t) \\
&= \mathcal{F}^{-1}\{\mathcal{F}\{\operatorname{sinc}(t)\}(f) \cdot \mathcal{F}\{\operatorname{sinc}(t)\}(f)\}(t) \\
&= \mathcal{F}^{-1}\{\operatorname{rect}(f) * \operatorname{rect}(f)\}(t) = \mathcal{F}^{-1}\{\operatorname{rect}(f)\}(t) \\
&= \operatorname{sinc}(t)
\end{aligned}$$

   In the case of the unnormalized sinc function, its Fourier transform would be a different rect function, which leads to a similar answer (the only difference being that there would be a non-unity scaling factor introduced when multiplying the different rect function with itself).

4. *Suppose a system acts on signal x and produces output y by the rule*

$$y(t) = |x(t+3)|$$

Is the system linear? Is the system time invariant? Is the system causal?

The system is not causal because it depends on future values of t (e.g., $y(0)$ is a function of $x(3)$). The system is not linear and is time-invariant:

$$\begin{aligned} y\{cx_1 + x_2\}(t) &= |cx_1(t+3) + x_2(t+3)| \\ &\neq c|x_1(t+3)| + |x_2(t+3)| = cy\{x_1\}(t) + y\{x_2\}(t) \end{aligned}$$

$$y\{x(t)\}(t - t_0) = |x((t - t_0) + 3)| = |x((t+3) - t_0)| = y\{x(t - t_0)\}(t)$$

5. *Let $x(t)$ be the signal given by $\cos(2\pi ft)$ for $0 \le t \le T$ and 0 otherwise, where $T = 1/f$. Find the Fourier transform, $X(\omega)$, of this signal. Demonstrate Parseval's theorem by comparing the norms of x and X.*

$$\begin{aligned} \mathcal{F}\{\cos(2\pi f_0 t)\}(f) &= \frac{1}{2}(\delta(f + f_0) + \delta(f - f_0)) \\ \mathcal{F}\{\mathrm{rect}_{(0,T)}(t)\}(f) &= \frac{1}{f_0} e^{-j\pi f/f_0} \operatorname{sinc} \frac{f}{f_0} \\ X(f) = \mathcal{F}\{x(t)\}(f) &= \mathcal{F}\{\cos(2\pi f_0 t) \cdot \mathrm{rect}_{(0,T)}(t)\}(f) \\ &= \mathcal{F}\{\cos(2\pi f_0 t)\}(f) * \mathcal{F}\{\mathrm{rect}_{(0,T)(t)}(f)\} \\ &= \frac{1}{2f_0}\left(e^{-j\pi(f+f_0)/f_0} \operatorname{sinc} \frac{f+f_0}{f_0} + e^{-j\pi(f-f_0)/f_0} \operatorname{sinc} \frac{f-f_0}{f_0}\right) \end{aligned}$$

Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Note that the support of x is $0 \le t \le T$ and the support of X is $-f_s/2 \le f \le f_s/2$, so we only have to integrate over these intervals.

$$\int_0^T |x(t)|^2 \, dt = \int_{-f_s/2}^{f_s/2} |X(f)|^2 \, df$$

I was able to calculate the energy of the signal analytically ($\int_0^T |x(t)|^2 \, dt = T/2$), but the integration of the latter was messy. Using MATLAB, I was able to check that the right hand side of the equation was also equal to $T/2$ by numerical integration (using `trapz`). With the parameters given in (6), the value of both integrals is equal to $T/2 = 0.25$. (Note: if we had integrated w.r.t. ω rather than f, there would be a scaling factor of $\frac{1}{2\pi}$.)

6. *Suppose we pass $x(t)$ from the previous question through the system from question 4. Use MATLAB to find the amplitude and phase of the output signal's Fourier transform $Y(\omega)$. Plot $X(\omega)$ as well – how do the signals compare?*

Parameters used:

- N (samples) $= 1000$

- T (sample window) $= 10$
- f_s (sampling frequency) $= N/T = 100$
- f_0 (cosine frequency) $= 2$
- T_0 (cosine period) $= f_0^{-1} = 0.5$

See (Figure 1) for the plots. Comparison of plots:

- As expected from the convolution of a sinc wave with two shifted deltas, $|X(f)|$ looks like it has has symmetric sinc-like waves centered at $\approx \pm f_0$. $|X(0)| = 0$, since there is no DC offset ($x(t)$ has an average value of 0).
- The plot of $|Y(f)|$ has a peak at $f = 0$ (DC), which makes sense now that the signal is purely positive. It has its two next-highest peaks at $\approx \pm 2f_0$, since $|\cos(\omega t)|$ looks somewhat like a sinusoid with double the frequency.
- The phase has changed from linearly increasing in $X(f)$ to linearly decreasing in $Y(f)$.

7. *Let $z(t) = y(t)cos(64\pi t + \theta)$. Write z as a sum of in-phase and quadrature components. Plot the Fourier transform $Z(\omega)$ for $\theta = \pi/3$, and comment on the effect of the modulation on amplitude and phase (compared to $Y(\omega)$.*

$$
\begin{aligned}
z(t) &= y(t)\cos(64\pi t + \theta) \\
&= y(t)(\cos(\theta)\cos(64\pi t) - \sin(theta)\sin(64\pi t)) \\
&= [y(t)\cos\theta]\cos(64\pi t) - [y(t)\sin\theta]\sin(64\pi t) \\
&= z_i(t)\cos(64\pi t) - z_q(t)\sin(64\pi t)
\end{aligned}
$$

(In phasor form, this is equivalent to $Z = [y(t)\cos\theta] + j[y(t)\sin\theta]$.)

(See (Figure 1) for the plots of $z(t)$ and $Z(f)$.) Notes on amplitude and phase:

- The magnitude plot of $|Z(f)|$ appears like the magnitude plot of $Y(f)$, duplicated with one centered at $\pm$32Hz (the carrier frequency, as expected).
- The magnitude values in $|Y(f)|$ are about twice the values of the corresponding waveforms in $|Z(f)|$.
- There is hardly any noticeable change in the phase plot.

8. *Write a function that takes as an input a time-domain signal and outputs the Hilbert transform of that signal (also in the time-domain). Plot the Hilbert transforms of x, y and z in the frequency and time domains.*

```
function res = hilbertTransform(x)
        X = fft(x);
        len = size(x, 2);
        % get signum of frequency;
        % +1 for (0, pi), -1 for (-pi, 0)
        sgn = [ones(1,floor(len/2), -1*ones(1,ceil(len/2))];
        res = ifft(-1j * sgn .* X);
end
```

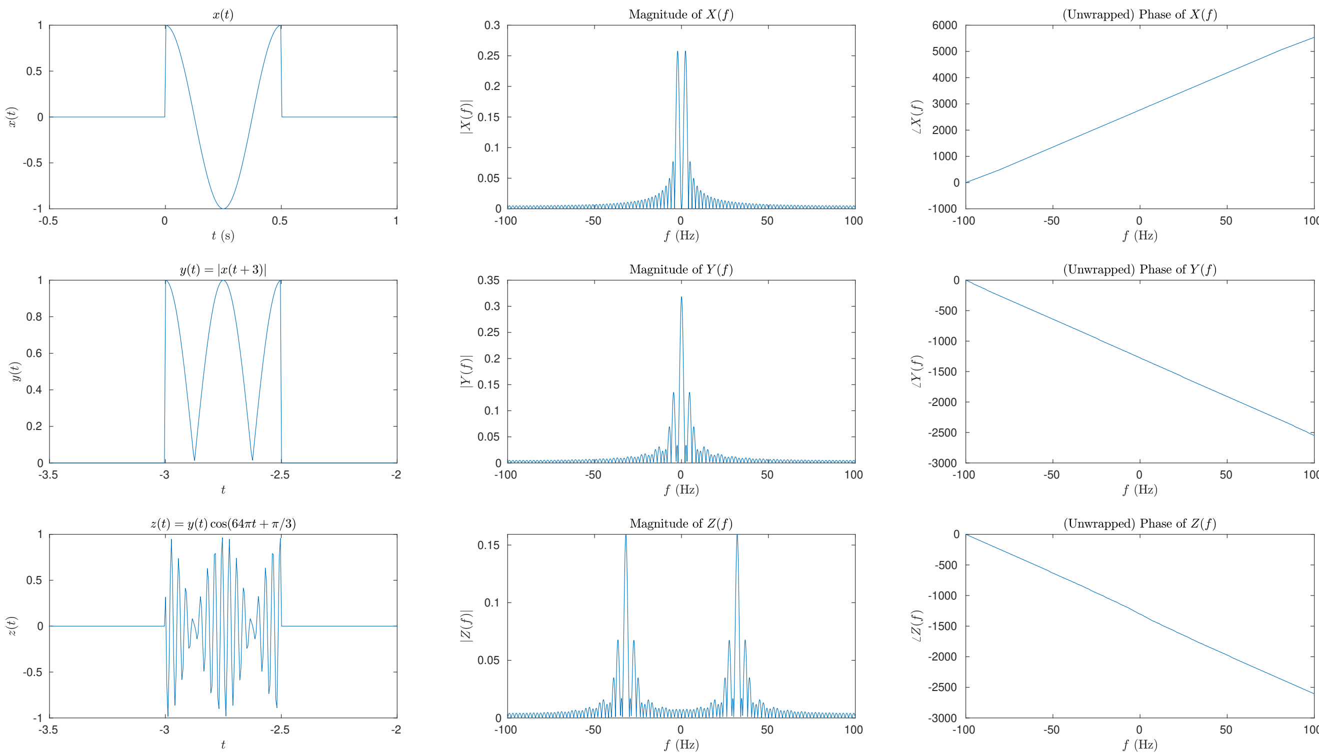

Figure 1: Plots of $x(t)$, $y(t)$, $z(t)$, and their Fourier transforms. See the parameters in (6).

See (Figure 2) for plots of the signals and their Hilbert transforms.

9. *Using MATLAB, demonstrate the orthogonality of a signal and its Hilbert transform for all of x, y and z.*

   `trapz` was used to estimate the integral for the inner product:

$$\langle x_1, x_2 \rangle = \int_{-\infty}^{\infty} x_1(t) x_2^*(t)\, dt$$

   The limits for t were $[-T, T]$, as it was for the previous parts of this problem set. As expected, the inner products were all close to zero:

$$|\langle x, \hat{x} \rangle| \approx 6.255 \times 10^{-9}$$
$$|\langle y, \hat{y} \rangle| \approx 1.014 \times 10^{-2}$$
$$|\langle z, \hat{z} \rangle| \approx 1.624 \times 10^{-8}$$

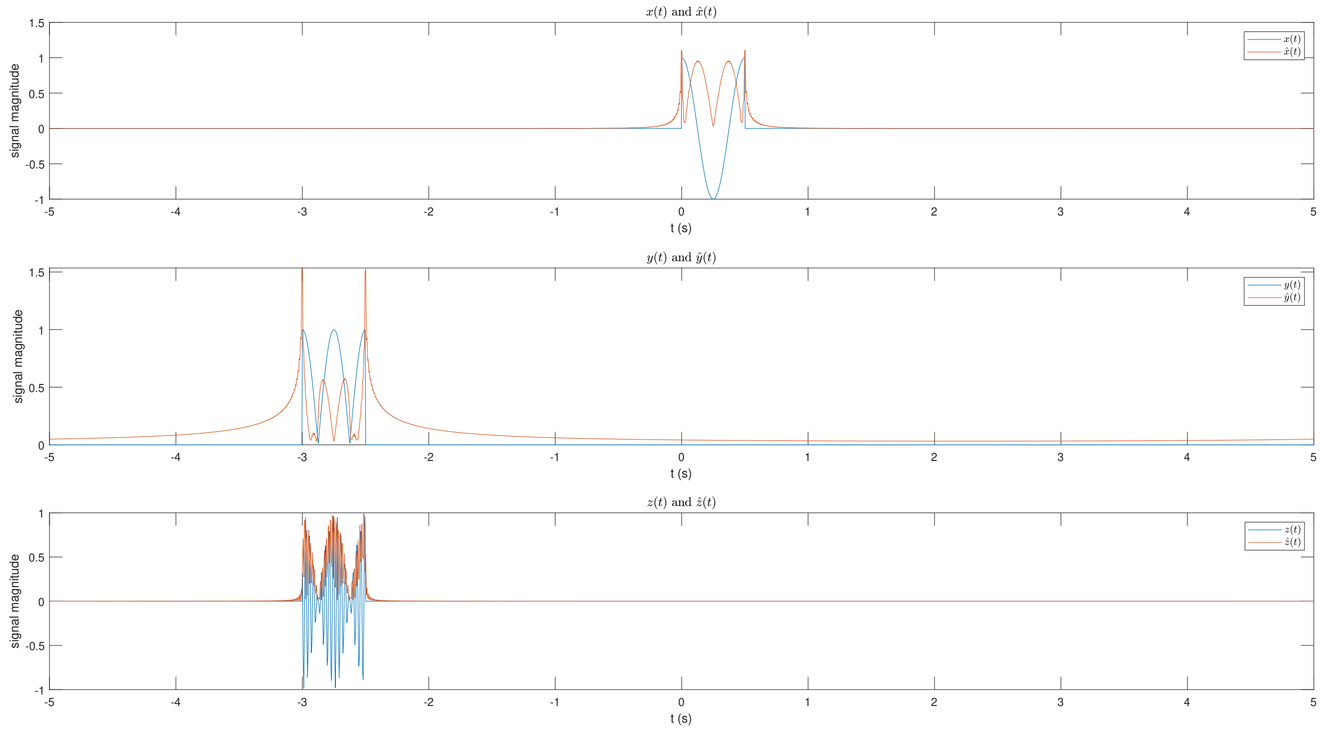

Figure 2: Plots of $x(t)$, $y(t)$, $z(t)$, and their Hilbert transforms.

Full MATLAB code:

```
% PSET1
% Jonathan Lam
% Prof. Frost
% ECE300
% Communications Theory
% 9/7/20

clear; close all; clc;
set(0, 'defaultTextInterpreter', 'latex');

% Q5

% sample details
N = 2000;
T = 10;
```

```
fs = N/T;

% sinusoid details
f0 = 2;
T0 = 1/f0;

% generate x(t)
t = linspace(-T/2, T/2, N);
x1 = cos(2*pi*f0*t);
x2 = rectangularPulse(0, T0, t);
x = x1 .* x2;

% plot x(t) from -T0 to 2*T0
figure();
subplot(3, 3, 1);
plot(t, x);
xlabel('$t$ (s)');
ylabel('$x(t)$');
title('$x(t)$');
xlim([-T0, 2*T0]);

% plot X(f) (magnitude and phase) for entire Nyquist bandwidth
X = fftshift(fft(x)/fs);
wd = linspace(-pi, pi, N);
f = wd * fs / (2 * pi);
subplot(3, 3, 2);
plot(f, abs(X));
xlabel('$f$ (Hz)');
ylabel('$|X(f)|$');
title('Magnitude of $X(f)$');
subplot(3, 3, 3);
plot(f, unwrap(angle(X)));
xlabel('$f$ (Hz)');
ylabel('$\angle X(f)$');
title('(Unwrapped) Phase of $X(f)$');

% compare x norm, X norm (they are equal)
% these both print out 0.25 (T/2) for the given parameters
fprintf('||x||=%d\n||X||=%d\n', trapz(t, abs(x).^2), trapz(f, abs(X).^2));

% Q6

% recompute x, shifted, and compute and plot y(t) from -3-T0 to -3+2*T0
x1Shifted = cos(2*pi*f0*(t+3));
x2Shifted = rectangularPulse(0, T0, (t+3));
xShifted = x1Shifted .* x2Shifted;
```

```
y = abs(xShifted);
subplot(3, 3, 4);
plot(t, y);
xlabel('$t$');
ylabel('$y(t)$');
title('$y(t)=|x(t+3)|$');
xlim([-3-T0, -3+2*T0]);

% plot Y(f) (magnitude and phase) for entire Nyquist bandwidth
Y = fftshift(fft(y)/fs);
subplot(3, 3, 5);
plot(f, abs(Y));
xlabel('$f$ (Hz)');
ylabel('$|Y(f)|$');
title('Magnitude of $Y(f)$');
subplot(3, 3, 6);
plot(f, unwrap(angle(Y)));
xlabel('$f$ (Hz)');
ylabel('$\angle Y(f)$');
title('(Unwrapped) Phase of $Y(f)$');

% Q7

% compute z(t) and plot on same x-axis as y(t)
z = y .* cos(64*pi*t + pi/3);
subplot(3, 3, 7);
plot(t, z);
xlabel('$t$');
ylabel('$z(t)$');
title('$z(t)=y(t)\cos(64\pi t+\pi/3)$');
xlim([-3-T0, -3+2*T0]);

% compute and plot Z(f)
Z = fftshift(fft(z)/fs);
subplot(3, 3, 8);
plot(f, abs(Z));
xlabel('$f$ (Hz)');
ylabel('$|Z(f)|$');
title('Magnitude of $Z(f)$');
subplot(3, 3, 9);
plot(f, unwrap(angle(Z)));
xlabel('$f$ (Hz)');
ylabel('$\angle Z(f)$');
title('(Unwrapped) Phase of $Z(f)$');

% Q8-9
```

```
innerProd = @(t, f, g) trapz(t, f .* conj(g));
figure();
signals = [x; y; z];
labels = ['x' 'y' 'z'];
for i = 1:3
    signal = signals(i, :);
    subplot(3, 1, i);
    plot(t, signal, t, abs(hilbertTransform(signal)));
    xlabel('t (s)');
    ylabel('signal magnitude');
    legend([string(sprintf('$%c(t)$', labels(i))), ...
        string(sprintf('$\\hat{%c}(t)$', labels(i)))], ...
        'interpreter', 'latex');
    title(sprintf('$%c(t)$ and $\\hat{%c}(t)$', labels(i), labels(i)), ...
        'interpreter', 'latex');

    % this should be fairly small (approximately 0) because the signals
    % and their hilbert transforms should be orthogonal
    fprintf('|<%c(t),\\hat{%c}(t)>|=%d\n', labels(i), labels(i), ...
        abs(innerProd(t, signal, hilbertTransform(signal))))
end

% Q8
function res = hilbertTransform(x)
    X = fft(x);
    len = size(x, 2);
    % get signum of frequency; +1 for 0 to pi, -1 for -pi to 0
    sgn = [ones(1, floor(len/2)), -1*ones(1, ceil(len/2))];
    res = ifft(-1j * sgn .* X);
end
```

MATLAB text output:

```
||x||=2.502502e-01
||X||=2.502502e-01
|<x(t),\hat{x}(t)>|=6.255166e-09
|<y(t),\hat{y}(t)>|=1.014327e-02
|<z(t),\hat{z}(t)>|=1.624092e-08
```