# PH214C – Pset 8

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## 1 P-current

*Using the definition for probability density $P = |\psi^2| = \psi^*\psi$, calculate $\partial P/\partial t$ to find that*

$$\frac{\partial P}{\partial t} + \frac{\partial \mathbb{J}}{\partial x} = 0$$

*in this way defining the probability current $\mathbb{J}$.*

We know that $\mathbb{J}$ is defined as:

$$\mathbb{J} \coloneqq \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

So:

$$\begin{aligned}
\frac{\partial}{\partial t}P &= \frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}\psi^*\psi = \psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t} \\
&= \psi^*\left[\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right)\right] + \psi\left[-\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right)\right] \\
&= \psi^*\left[-\frac{\hbar}{2mi}\frac{\partial^2\psi}{\partial x^2}\right] + \frac{V}{i\hbar}\psi^*\psi + \psi\left[\frac{\hbar}{2mi}\frac{\partial^2\psi}{\partial x^2}\right] - \frac{V}{i\hbar}\psi^*\psi \\
&= \frac{i\hbar}{2m}\left[\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right] \\
&= \frac{i\hbar}{2m}\left[\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} + \psi^*\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)\right) - \left(\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x} + \psi\frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}\right)\right)\right] \\
&= \frac{i\hbar}{2m}\left[\frac{\partial}{\partial x}\left(\psi^*\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\psi\frac{\partial\psi^*}{\partial x}\right)\right] \\
&= \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)\right] \\
&= -\frac{\partial}{\partial x}\mathbb{J} \\
&\Rightarrow \frac{\partial P}{\partial t} + \frac{\partial\mathbb{J}}{\partial x} = 0
\end{aligned}$$

## 2 0-current

*For the potential step problem, in the case that $E < V_0$, there is a finite probability that a particle is found in the forbidden region, $x > 0$, where $E < V_0$. Yet the probability current in the transmitted region, $\mathbb{J}_{trans} = 0$. Show these results.*

The time-independent solution $U$ to the finite step function was governed by the following equations. Note that $U_L$ denotes the time-independent solution to the Schödinger equation for $x < 0$ ($V(x) = 0$), and $U_R$ denotes the like for $x > 0$ ($V(x) = V_0$).

$$\begin{aligned} \frac{d^2U_L}{dx^2} &= -\frac{2mE}{\hbar^2}U_R \\ U_L(x) &= e^{ikx} + Re^{-ikx} \\ k^2 &= \frac{2mE}{\hbar^2} \\ R &= \frac{k-\kappa}{k+\kappa} \end{aligned} \qquad \begin{aligned} \frac{d^2U_R}{dx^2} &= -\frac{2m(E-V_0)}{\hbar^2}U_R \\ U_R(x) &= Te^{i\kappa x} \\ \kappa^2 &= \frac{2m(E-V_0)}{\hbar^2} \\ T &= \frac{2k}{k+\kappa} \end{aligned}$$

We are interested in the solution in the positive region, where $E < V_0$. This would suggest that $\kappa$ becomes (pure) imaginary, so that:

$$\kappa = i\frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$U_R = T\exp\left(-\frac{2m(V_0 - E)}{\hbar}x\right)$$

This is a decaying exponential with coefficient $T \neq 0$, so $U_R(x) \neq 0$ in the forbidden region. Since $S(t)$ is also an exponential (and thus not identically zero), then $P = |\psi^2| = |U(x)S(t)| \neq 0$, there is a nonzero probability that a particle is found in the forbidden region. However, if we were to calculate the probability current, it becomes zero now that $U_R$ becomes pure real:

$$\begin{aligned} \mathbb{J}_R &= \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right) \\ &= \frac{i\hbar}{2m}\left([U_R(x)S(t)]\frac{\partial}{\partial x}[U_R(x)S^*(t)] - [U_R(x)S^*(t)]\frac{\partial}{\partial x}[U_R(x)S(t)]\right) \\ &= \frac{i\hbar}{2m}\left(U_RSS^*\frac{\partial U_R}{\partial x} - U_RS^*S\frac{\partial U_R}{\partial x}\right) \\ &= \frac{i\hbar}{2m}(0) \\ &= 0 \end{aligned}$$

In other words, the probability distribution of the particles in the forbidden region is not evolving over time.

## How low can you go?

*Use the property of the lowering operator and ground state of the QMSHO:*

$$\hat{a}^{-}\psi_0 = 0$$

*to find the ground state wave function* $\psi_0$*. Now raise* $\psi_0$ *to get* $\psi_1$ *and then lower* $\psi_1$ *back to* $\psi_0$*. Do you get back to your starting point or is there a "leftover"?*

Definitions:

$$\hat{a}^{-} := \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$
$$\hat{a}^{+} := \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{p} := -i\hbar\frac{\partial}{\partial x}$$

Solving the ODE:

$$\begin{aligned}
\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\psi_0 &= 0 \\
\frac{\hbar}{m\omega}\frac{\partial\psi_0}{\partial x} + x\psi_0 &= 0 \\
\frac{d\psi_0}{\psi_0} &= -\frac{m\omega x}{\hbar}\,dx \\
\psi_0 &= A\exp\left(-\frac{m\omega x^2}{2\hbar}\right)
\end{aligned}$$

Normalizing:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} dx\,\psi^*\psi \\
&= \int_{-\infty}^{\infty} dx\,A^*\exp\left(-\frac{m\omega x^2}{2\hbar}\right)A\exp\left(-\frac{m\omega x^2}{2\hbar}\right) \\
&= |A|^2\int_{-\infty}^{\infty} dx\,\exp\left(-\frac{m\omega x^2}{\hbar}\right) \\
|A| &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
\psi_0 &= \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)
\end{aligned}$$

To save some typing, let $\alpha = m\omega\hbar^{-1}$. "Raising" $\psi_0$ to $\psi_1'$ (non-normalized):

$$\begin{aligned}
\psi_1' &= \hat{a}^+\psi_0 \\
&= \sqrt{\frac{\alpha}{2}}\left(\hat{x} + \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\psi_0 \\
&= \sqrt{\frac{\alpha}{2}}\left(\hat{x} - \frac{1}{\alpha}\frac{\partial}{\partial x}\right)\left[\sqrt[4]{\frac{\alpha}{\pi}}\exp\left(-\frac{\alpha x^2}{2}\right)\right] \\
&= \sqrt{\frac{\alpha}{2}}\sqrt[4]{\frac{\alpha}{\pi}}\left(x\exp\left(-\frac{\alpha x^2}{2}\right) - \frac{1}{\alpha}\left(-\frac{2\alpha x}{2}\right)\exp\left(-\frac{\alpha x^2}{2}\right)\right) \\
&= \sqrt{\frac{\alpha}{2}}\sqrt[4]{\frac{\alpha}{\pi}}(2x)\exp\left(-\frac{\alpha x^2}{2}\right) \\
&= \sqrt{2\alpha}x\psi_0
\end{aligned}$$

"Lowering" $\psi_1'$ to $\psi_0'$:

$$\begin{aligned}
\psi_0' &= \hat{a}^-\psi_1' \\
&= \sqrt{\frac{\alpha}{2}}\left(\hat{x} - \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\psi_1' \\
&= \sqrt{\frac{\alpha}{2}}\sqrt{2\alpha}\sqrt[4]{\frac{\alpha}{\pi}}\left(\hat{x} + \frac{1}{\alpha}\frac{\partial}{\partial x}\right)\left[x\exp\left(-\frac{\alpha x^2}{2}\right)\right] \\
&= \alpha\sqrt[4]{\frac{\alpha}{\pi}}\left(x^2\exp\left(-\frac{\alpha x^2}{2}\right) + \frac{1}{\alpha}\left(-\frac{2\alpha x}{2}x\exp\left(-\frac{\alpha x^2}{2}\right) + \exp\left(-\frac{\alpha x^2}{2}\right)\right)\right) \\
&= \alpha\sqrt[4]{\frac{\alpha}{\pi}}\left(\frac{1}{\alpha}\right)\exp\left(-\frac{\alpha x^2}{2}\right) \\
&= \psi_0
\end{aligned}$$

We see that $\psi_0' = \psi_0$, so there is no "leftover" scaling factor. Or, if we know the scaling factor caused by the raising and lowering operators (shown in the notes):

$$\begin{aligned}
\hat{a}^+\psi_n &= \sqrt{n+1}\psi_{n+1} \\
\hat{a}^-\psi_n &= \sqrt{n}\psi_{n-1}
\end{aligned}$$

then we get the same result:

$$\hat{a}^-\hat{a}^+\psi_0 = \hat{a}^-(\sqrt{0+1}\psi_{0+1}) = \hat{a}^-\psi_1 = \sqrt{1}\psi_{1-1} = \psi_0$$

# Operator, operator

*Find the position and momentum operators, $\hat{x}$ and $\hat{p}$, for the SHO potential in terms of the raising and lowering operators, $\hat{a}^+$ and $\hat{a}^-$.*

We defined the raising and lowering operators partially in terms of $\hat{p}$ and $\hat{x}$, so we just need to solve in reverse. Solving for $\hat{p}$:

$$\hat{a}^- - \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(2\frac{i\hat{p}}{m\omega}\right) = \sqrt{\frac{2}{m\omega\hbar}}i\hat{p}$$

$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^- - \hat{a}^+)$$

Solving for $\hat{x}$:

$$\hat{a}^- + \hat{a}^- = \sqrt{\frac{m\omega}{2\hbar}}(2\hat{x}) = \sqrt{\frac{2m\omega}{\hbar}}\hat{x}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^- + \hat{a}^+)$$

## Great expectations

*Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, and $\langle K\rangle$ (where $K$ is the kinetic energy) for the $n$-th state of the SHO.*

We use the linearity of the bra-ket (expectation value, or inner product) and the orthonormality of solutions ($\langle\psi_n|\psi_m\rangle = \delta_{nm}$) implicitly.

$$\begin{aligned}
\langle x\rangle_n &= \langle\psi_n|\hat{x}\psi_n\rangle \\
&= \left\langle\psi_n\middle|\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^- + \hat{a}^-)\psi_n\right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle\psi_n|\hat{a}^-\psi_n\rangle + \langle\psi_n|\hat{a}^+\psi_n\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\langle\psi_n|\psi_{n-1}\rangle + \sqrt{n+1}\,\langle\psi_n|\psi_{n+1}\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}(0) + \sqrt{n+1}(0)\right] \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\langle p \rangle_n &= \langle \psi_n | \hat{p} \psi_n \rangle \\
&= \left\langle \psi_n \middle| -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^- - \hat{a}^+)\psi_n \right\rangle \\
&= -i\sqrt{\frac{m\omega\hbar}{2}} \left[ \langle \psi_n | \hat{a}^- \psi_n \rangle - \langle \psi_n | \hat{a}^+ \psi_n \rangle \right] \\
&= i\sqrt{\frac{m\omega\hbar}{2}} \left[ \sqrt{n} \langle \psi_n | \psi_{n-1} \rangle - \sqrt{n+1} \langle \psi_n | \psi_{n+1} \rangle \right] \\
&= i\sqrt{\frac{m\omega\hbar}{2}} \left[ \sqrt{n}(0) - \sqrt{n+1}(0) \right] \\
&= 0
\end{aligned}
$$

Intuitively, the expectations for position and momentum can be seen by the symmetry of the SHO potential and an intuitive understanding of displacement and momentum in the classical sense. The variances are less obvious.

$$
\begin{aligned}
\langle x^2 \rangle_n &= \langle \psi_n | \hat{x}^2 \psi_n \rangle \\
&= \left\langle \psi_n \middle| \frac{\hbar}{2m\omega}(\hat{a}^- + \hat{a}^+)^2 \psi_n \right\rangle \\
&= \frac{\hbar}{2m\omega} \left[ \left\langle \psi_n \,\middle|\, (\hat{a}^-)^2 \psi_n \right\rangle + \left\langle \psi_n \,\middle|\, \hat{a}^- \hat{a}^+ \psi_n \right\rangle + \left\langle \psi_n \,\middle|\, \hat{a}^+ \hat{a}^- \psi_n \right\rangle + \left\langle \psi_n \,\middle|\, (\hat{a}^+)^2 \psi_n \right\rangle \right] \\
&= \frac{\hbar}{2m\omega} \left[ \sqrt{n(n-1)} \langle \psi_n | \psi_{n-2} \rangle + (n+1) \langle \psi_n | \psi_n \rangle + n \langle \psi_n | \psi_n \rangle + \sqrt{(n+1)(n+2)} \langle \psi_n | \psi_{n+2} \rangle \right] \\
&= \frac{\hbar}{2m\omega} \left[ \sqrt{n(n-1)}(0) + (n+1)(1) + n(1) + \sqrt{(n+1)(n+2)}(0) \right] \\
&= \frac{\hbar}{2m\omega}(2n+1)
\end{aligned}
$$

$$
\begin{aligned}
\langle p^2 \rangle_n &= \langle \psi_n | \hat{p}^2 \psi_n \rangle \\
&= \left\langle \psi_n \middle| -\frac{m\omega\hbar}{2}(\hat{a}^- - \hat{a}^+)\psi_n \right\rangle \\
&= -\frac{m\omega\hbar}{2} \left[ \left\langle \psi_n \,\middle|\, (\hat{a}^-)^2 \psi_n \right\rangle - \left\langle \psi_n \,\middle|\, \hat{a}^- \hat{a}^+ \psi_n \right\rangle - \left\langle \psi_n \,\middle|\, \hat{a}^+ \hat{a}^- \psi_n \right\rangle + \left\langle \psi_n \,\middle|\, (\hat{a}^+)^2 \psi_n \right\rangle \right] \\
&= -\frac{m\omega\hbar}{2} \left[ \sqrt{n(n-1)}(0) - (n+1)(1) - n(1) + \sqrt{(n+1)(n+2)}(0) \right] \\
&= \frac{m\omega\hbar}{2}(2n+1)
\end{aligned}
$$

$$
\langle K \rangle_n = \left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2m} \langle p^2 \rangle = \frac{\omega\hbar}{4}(2n+1)
$$