PH214C – Pset 6

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1 A Bohr result

Suppose that an integer number of de Broglie wavelengths fit around a circular Bohr orbit ($n \in \mathbb{Z}_+$). Find an expression for the quantized angular momentum for general n and sketch the case for $n = 4$.

$$l = mvr \qquad \text{(angular momentum)}$$

$$\lambda = \frac{h}{mv} \qquad \text{(de Broglie wavelength)}$$

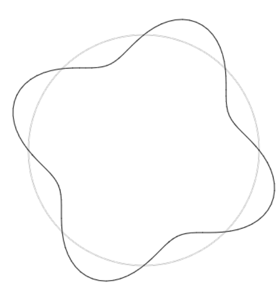
$$2\pi r = n\lambda \qquad \text{(quantized orbit)}$$

Thus

$$l = \frac{h}{\lambda} r = \frac{hn}{2\pi r} r = n\hbar$$

In the case where $n = 4$,

$$l_4 = 4\hbar = 4.05 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}$$



2 Wave packet

Show that our definition of the wave function of a wave packet

$$\psi(x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\,\phi(p)e^{i(px-Et)/\hbar}$$

solves the free-particle Schödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

but not the generic wave equation:

$$\frac{\partial^2\psi}{\partial t^2} = a^2\frac{\partial^2\psi}{\partial x^2}$$

2.1 Free-particle Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\left(-\frac{iE}{\hbar}\psi\right) \qquad = E\psi \qquad \text{(LHS)}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = -\frac{\hbar^2}{2m}\left(\left(\frac{ip}{\hbar}\right)^2\psi\right) = \frac{p^2}{2m}\psi \qquad = E\psi \qquad \text{(RHS)}$$

2.2 Generic wave equation

$$\frac{\partial^2\psi}{\partial t^2} = \left(\frac{-iE}{\hbar}\right)^2\psi \qquad = -\frac{E^2}{\hbar^2}\psi \qquad \text{(LHS)}$$

$$a^2\frac{\partial^2\psi}{\partial x^2} = a^2\left(\frac{ip}{\hbar}\right)^2\psi \qquad = -\frac{a^2p^2}{\hbar^2}\psi \qquad \text{(RHS)}$$

For ψ to satisfy the wave equation, both sides must be equal:

$$-\frac{E^2}{\hbar^2}\psi = -\frac{a^2p^2}{\hbar^2}\psi$$

$$E^2 \propto p^2$$

$$\text{but } E = \frac{p^2}{2m} \Rightarrow \text{contradiction}$$

3 Probability

We found the peak probability and expectation value $\langle x\rangle$ of the wave function

$$\psi(x) = 2a^{3/2}xe^{-ax}, \text{ for } x > 0, a = \text{constant}$$

Verify these results and then find the probability that the particle will be found between $x = 0$ and $x = 1/a$.

3.1 Peak probability

This occurs when $P = |\psi|^2$ is maximized. (Assume $a > 0$.)

$$\begin{aligned} P(x) &= |\psi(x)|^2 = 4a^3x^2e^{-2ax} \\ \frac{dP}{dx} &= 8a^3x(1-ax)e^{-2ax} = 0 \\ \frac{d^2P}{dx^2} &= 8a^3(1-4ax+2a^2x^2)e^{-2ax} \end{aligned}$$

This equation is satisfied when $x = 0$ or $x = 1/a$. Using the second derivative:

$$\begin{aligned} \left.\frac{d^2P}{dx}\right|_{x=0} &= 8a^3 > 0 \\ \left.\frac{d^2P}{dx}\right|_{x=1/a} &= -8a^3e^{-2} < 0 \end{aligned}$$

Thus the only local maximum occurs at $x = 1/a$. (Also, since the probability dies out as $x \to \infty$, this must be the global maximum in the interval $[0, \infty)$.)

3.2 Expectation value

Since $\psi = 0$ for negative x, we can begin the integration at $x = 0$. Since ψ and x are real, $P = |\psi|^2 = \psi^*\psi = \psi^2$, and $\psi^*x\psi = x\psi^2 = xP$.

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} dx\, \psi^* x \psi \\ &= \int_0^{\infty} dx\, x(4a^3x^2e^{-2ax}) \\ &= \frac{1}{4a}\int_0^{\infty} dy\, y^3e^{-y} && (y = 2ax) \\ &= \frac{1}{4a}\left[e^{-y}(-y^3-3y^2-6y-6)\right|_{x=0}^{\infty} && \text{(a lot of integration by parts)} \\ &= \frac{3}{2a} \end{aligned}$$

3.3 Probability of position between 0 and $1/a$

This is just an integral of probability over some interval of x.

$$\begin{aligned}
P\left(0 \le x \le \frac{1}{a}\right) &= \int_0^{1/a} dx\, P(x) \\
&= \int_0^{1/a} dx\, (4a^3 x^2 e^{-2ax}) \\
&= \frac{1}{2}\int_0^2 dy\, y^2 e^{-y} && (y = 2ax) \\
&= \frac{1}{2}\left[e^{-y}(-y^2 - 2y - 2)\right|_0^2 && \text{(integration by parts)} \\
&= \frac{1}{2}(e^{-2}(-4-4-2) - e^0(-2)) \\
&= 1 - 5e^{-2} \approx 0.32
\end{aligned}$$

4 Momentum keeps it real

Show that $\langle p\rangle - \langle p\rangle^* = 0$ to show that the operator for p is Hermitian.

$$\hat{p} := \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\begin{aligned}
\langle p\rangle - \langle p\rangle^* &= \left[\int_{-\infty}^{\infty} dx\, \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi\right] - \left[\int_{-\infty}^{\infty} dx\, \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi\right]^* \\
&= \frac{\hbar}{i}\left[\int_{-\infty}^{\infty} dx\, \psi^* \frac{\partial \psi}{\partial x}\right] - \left(\frac{\hbar}{-i}\right)\left[\int_{-\infty}^{\infty} dx\, \psi \frac{\partial \psi^*}{\partial x}\right] \\
&= \frac{\hbar}{i}\int_{-\infty}^{\infty} dx\, \psi^* \frac{\partial \psi}{\partial x} + \psi \frac{\partial \psi^*}{\partial x} \\
&= \frac{\hbar}{i}\int_{-\infty}^{\infty} dx\, \frac{\partial}{\partial x}(\psi^* \psi) = \frac{\hbar}{i}\int_{-\infty}^{\infty} dx\, \frac{\partial |\psi|^2}{\partial x} \\
&= \frac{\hbar}{i}\lim_{b\to\infty} |\psi|^2\Big|_{x=-b}^{b} = \frac{\hbar}{i}\,[0 - 0] = 0
\end{aligned}$$

Notes about the math:

- Any valid PDF (e.g., $|\psi|^2$) must approach the value 0 in the limit as the variable goes to $\pm\infty$.
- In general,

  $$\left[\int f(x)\, dx\right]^* = \int [f(x)]^*\, [dx]^*$$

  Since x and dx are real, then the conjugate of the integral is the integral of the conjugate of the integrand.