Pset 5 – PH214C

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Don't box me in *We wish to derive the final coefficient in the classical formula for energy density,*

$$u = kT\left(\frac{8\pi\nu^2}{c^3}\right)$$

The quantity we are trying to find is the density of states w.r.t. frequency and space. In other words, if N represents number of states, then we are trying to find

$$\frac{\partial^2 N}{\partial V \partial \nu}$$

The modes must all have nodes in the x, y, and z directions. Since these nodes are stationary, the EM wave must be a standing wave:

$$\vec{E} = \vec{E}_0 \sin(k_x x)\sin(k_y y)\sin(k_z z)\sin(\omega t)$$

with the node imposing the condition (assume the box is a cube with length L):

$$\sin(k_x L) = 0 \Rightarrow k_x L = \pi n_x \ (n_x \in \mathbb{Z}_+)$$

in the x, y, and z directions. By substituting $k = \frac{2\pi}{\lambda} = \frac{2\pi c}{\nu}$,

$$\nu_x = n_x \frac{c}{2L} \ (n_x \in \mathbb{Z}_+)$$

This is a set of discrete frequencies in one direction. More generally, a three-dimensional mode can be uniquely defined by its tuple of (n_x, n_y, n_z) coordinates, and so we define this as the phase space. In this space, there is clearly one mode at every (discrete integer-tuple) coordinate (but only existing in the first octant), so the number density δ is:

$$\delta = \frac{1 \text{ mode}}{1 \text{ (unit volume)}}$$

where, if N is a number of modes and V is a (dimensionless) "volume" in this coordinate space, then

$$N = Vd$$

In other words, the number of modes is numerically equal to the volume in this space. Let

$$n \coloneqq \sqrt{n_x^2 + n_y^2 + n_z^2}$$

indicate the magnitude of the position vector in this space and

$$\nu \coloneqq \sqrt{\nu_x^2 + \nu_y^2 + \nu_z^2}$$

indicate the magnitude of the frequency. To find the "density" of this space with respect to frequency, we can find the number of modes with a frequency (magnitude) between ν_0 and $\nu_0 + d\nu$ and divide by $d\nu$ (i.e., differentiating w.r.t. frequency). The change in volume in the first octant is an eighth of a thin spherical shell with thickness ν. Since the number density is equal to the volume in the n coordinates, we begin by stating the volume of such a shell in n coordinates and then convert it to ν.

$$dN = dV = \frac{1}{8}\left(4\pi n^2\right)\, dn$$

$$n = \frac{2L}{c}\nu,\ dn = \frac{2L}{c}d\nu$$

$$\frac{dN}{d\nu} = \frac{4\pi L^3 \nu^2}{c^3}$$

Now, if we wish to find the density w.r.t. volume (in the regular coordinate system), we can divide by the volume of the cube, L^3. (It is a derivative in the limit of $L \to 0$.)

$$\frac{\partial^2 N}{\partial V \partial \nu} = \frac{4\pi\nu^2}{c^3}$$

We can follow a similar approach to find the analogue w.r.t. λ. Make the substitutions $\nu = c/\lambda$ and $d\nu = -c/\lambda^2\, d\lambda$ to arrive at the analogous density w.r.t. space and wavelength:

$$\frac{\partial^2 N}{\partial V\left(-\frac{c}{\lambda^2}\partial\lambda\right)} = \frac{4\pi\left(\frac{c}{\lambda}\right)^2}{c^3}$$

$$\frac{\partial^2 N}{\partial V \partial \lambda} = -\frac{4\pi}{\lambda^4}$$

Note that, in this derivative formulation, the "density" of modes w.r.t. λ is negative. To get the desired factor of 8, we note that there are actually two valid polarizations for each (n_x, n_y, n_z) tuple, so we double both of these results:

$$\frac{\partial^2 N}{\partial V \partial \nu} = \frac{8\pi\nu^2}{c^3}$$

$$\frac{\partial^2 N}{\partial V \partial \lambda} = -\frac{8\pi}{\lambda^4}$$

Certainly! *Reformulate the wave packet-based derivation in the notes so that we get, for the uncertainty product:*

$$\Delta k \Delta x \geq \frac{1}{2}$$

Using a Gaussian for $A(k)$ and the given measure of spread (the width of the Gaussian at height e^{-1}) gives the constant 4. Thus, if we reformulate the problem to use a different measure of spread (i.e., to standard deviation), we can change the constant that the uncertainties of wave number and position multiply to. Start with the same Gaussian wave number amplitude distribution:

$$A(k) = e^{-\frac{\alpha}{2}(k-k_0)^2}$$

The inverse Fourier transform of this wave number (at $t = 0$) is the waveform w.r.t. position (which is also a Gaussian amplitude distribution):

$$\Psi(x, 0) = \int_{-\infty}^{\infty} dk\, A(k) e^{ikx} = \sqrt{\frac{2\pi}{\alpha}} e^{ik_0 x} e^{-\frac{x^2}{2\alpha}}$$

(We neglect the tedious details of the inverse Fourier transform calculation here and use the result from the lecture notes.) Noting that the squares of A and Ψ form the PDF $f(k)$ of the wave number and PDF $g(x)$ of the position, respectively:

$$f(k) = A^2(k) = e^{-\alpha(k-k_0)^2}$$

$$g(x) = |\Psi^2(x, 0)| = \frac{2\pi}{\alpha} e^{-\frac{x^2}{\alpha}}$$

The general form of a Gaussian PDF is

$$h(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Since f and g are clearly in the form of a Gaussian (although not normalized, but this doesn't affect spread), we can see by matching the exponent to the Gaussian form that:

$$\sigma_k = \frac{1}{\sqrt{2\alpha}}$$

$$\sigma_x = \sqrt{\frac{\alpha}{2}}$$

Using this measure of spread instead of the Δ-spread defined in the lecture notes, we get:

$$\sigma_k \sigma_x = \sqrt{\frac{1}{2\alpha}} \sqrt{\frac{\alpha}{2}} = \frac{1}{2}$$

which saturates the stated lower bound of $\frac{1}{2}$. (This doesn't show the inequality, but we obtain a product lower than 4 from the lecture notes.)