PSET 4

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(Dielectric) Raindrops keep falling on my head

$$\phi_1 = -E_0 r\cos\theta + \frac{1}{4\pi\epsilon_1}\frac{A_1}{r^2}\cos\theta,\ \phi_2 = \frac{1}{4\pi\epsilon_2}B_1 r\cos\theta$$

(a) *Find the radial electric fields.*

$$E_1 = -\frac{\partial\phi_1}{\partial r} = E_0\cos\theta + \frac{1}{2\pi\epsilon_1}\frac{A_1}{r^3}\cos\theta$$

$$E_2 = -\frac{\partial\phi_2}{\partial r} = -\frac{1}{4\pi\epsilon_2}B_1\cos\theta$$

(b) *Find the unknown coefficients A_1 and B_1 by imposing the two matching conditions at $r = R$.*

Continuity of the normal D field:

$$\epsilon_1 E_1(R) = \epsilon_2 E_2(R)$$

$$\epsilon_1\left(E_0\cos\theta + \frac{1}{2\pi\epsilon_1}\frac{A_1}{R^3}\cos\theta\right) = \epsilon_2\left(-\frac{1}{4\pi\epsilon_2}B_1\cos\theta\right)$$

Continuity of the potential (continuity of the tangential E field):

$$\phi_1(R) = \phi_2(R)$$

$$-E_0 R\cos\theta + \frac{1}{4\pi\epsilon_1}\frac{A_1}{R^2}\cos\theta = \frac{1}{4\pi\epsilon_2}B_1 R\cos\theta$$

$$\begin{cases} \epsilon_1 E_0 + \frac{1}{2\pi}\frac{A_1}{R^3} = -\frac{\epsilon_2}{4\pi\epsilon_2}B_1 \\ -E_0 R + \frac{1}{4\pi\epsilon_1}\frac{A_1}{R^2} = \frac{1}{4\pi\epsilon_2}B_1 R_1 \end{cases}$$

$$\begin{cases} \epsilon_1 E_0 + \frac{1}{2\pi}\frac{A_1}{R^3} = -\frac{1}{4\pi}B_1 \\ -\epsilon_2 E_0 + \frac{\epsilon_2}{4\pi\epsilon_1}\frac{A_1}{R^3} = \frac{1}{4\pi}B_1 \end{cases}$$

$$(\epsilon_1 - \epsilon_2)E_0 + \frac{2\epsilon_1 + \epsilon_2}{4\pi\epsilon_1}\frac{A_1}{R^3} = 0$$

$$A_1 = \frac{4\pi\epsilon_1 R^3}{2\epsilon_1 + \epsilon_2}(\epsilon_2 - \epsilon_1)E_0$$

Plugging A_1 back into an earlier equation to solve for B_1:

$$\epsilon_1 E_0 + \frac{1}{2\pi}\frac{A_1}{R^3} = \epsilon_1 E_0 + \frac{2\epsilon_1 R^3}{2\epsilon_1 + \epsilon_2}(\epsilon_2 - \epsilon_1)E_0 = -\frac{1}{4\pi}B_2$$

$$B_1 = -4\pi\epsilon_1 E_0 + \frac{8\pi\epsilon_1 R^3}{2\epsilon_1 + \epsilon_2}(\epsilon_1 - \epsilon_2)E_0 = 4\pi\epsilon_1\left(\frac{2R^3}{2\epsilon_1 + \epsilon_2}(\epsilon_1 - \epsilon_2) - 1\right)E_0$$

...simplify a little...

$$B_1 = \frac{-12\pi\epsilon_1\epsilon_2 E_0}{2\epsilon_1 + \epsilon_2}$$

(c) *Show that A_1 is a dipole moment (the "effective dipole moment" of the sphere).*

$$\phi_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta$$

Looking at the voltage equation outside of the sphere, the second term (the voltage caused by the sphere and not the external electric force) is:

$$\phi_{sph} = \frac{1}{4\pi\epsilon_1}\frac{A_1}{r^2}\cos\theta$$

Thus, A_1 is analogous to p in a dipole.

$p_{eff} = KR^3E_0$. *Define the Clausius-Mossotti function:* $K = K(\epsilon_1, \epsilon_2)$ *What do we know about K?*

$$K = \frac{4\pi\epsilon_1}{2\epsilon_1 + \epsilon_2}(\epsilon_2 - \epsilon_1)$$

$\epsilon_2 > \epsilon_1 \Rightarrow K > 0$, and vice versa; and $\epsilon_1 = \epsilon_2 \Rightarrow K = 0$.

T and R *Based on derivations from lecture, we get the reflected and transmitted E field amplitudes at normal incidence:*

$$E_r = E_i\left(\frac{n_2 - n_1}{n_2 + n_1}\right),\ E_t = E_i\left(\frac{2n_1}{n_2 + n_1}\right)$$

(a) *Find the reflected and transmitted energy flux density $\vec{S}$ and then the time average $\langle\vec{S}\rangle$.*

(Non-vectorized fields indicate their magnitudes: e.g., $E_i = |\vec{E}_i|$)

$$E = Bv = \mu H\frac{1}{\sqrt{\mu\epsilon}} \Rightarrow H = \sqrt{\frac{\epsilon}{\mu}}E = \frac{1}{Z}E$$

$$\vec{S} = \vec{E}\times\vec{H} = EH\hat{k} = \frac{1}{Z}E^2\hat{k} = \frac{1}{Z}(\vec{E}\cdot\vec{E})\vec{k}$$

Reflected Poynting vector:

$$\vec{S}_r = \frac{1}{Z_1}\left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 E_i^2 \hat{k}, \ \langle \vec{S}_r \rangle = \frac{1}{2Z_1}\left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 E_{i_0}^2 \hat{k}$$

Transmitted Poynting vector:

$$\vec{S}_t = \frac{1}{Z_1}\left(\frac{2n_1}{n_2 + n_1}\right)^2 E_i^2 \hat{k}, \ \langle \vec{S}_t \rangle = \frac{1}{2Z_1}\left(\frac{2n_1}{n_2 + n_1}\right)^2 E_{i_0}^2 \hat{k}$$

(b) *Show that energy is conserved.*
To show this, need to show that $E_i = E_r + E_t$.

$$E_i = E_i \frac{n_2 - n_1 + 2n_1}{n_2 + n_1} = E_i \frac{n_2 - n_1}{n_2 + n_1} + E_i \frac{2n_1}{n_2 + n_1} = E_r + E_t$$

(c) *Compute the reflection and transmission coefficients, R and T, for power and show that $R + T = 1$.*

$$R = \frac{\langle S_r \rangle}{\langle S_i \rangle} = \frac{\frac{1}{2Z_1}\left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 E_{i_0}^2}{\frac{1}{2Z_1} E_{i_0}^2} = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2$$

$$T = \frac{\langle S_t \rangle}{\langle S_i \rangle} = \frac{\frac{1}{2Z_2}\left(\frac{2n_1}{n_2 + n_1}\right)^2 E_{i_0}^2}{\frac{1}{2Z_1} E_{i_0}^2} = \frac{Z_1}{Z_2}\left(\frac{2n_1}{n_2 + n_1}\right)^2 = \frac{n_2}{n_1}\left(\frac{2n_1}{n_2 + n_1}\right)^2$$

(Note that $n = c/v = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \approx \sqrt{\epsilon/\epsilon_0} \propto \sqrt{\epsilon/\mu} = Z^{-1}$.)

$$R + T = \frac{(n_2 - n_1)^2}{(n_2 + n_1)^2} + \frac{4n_1 n_2}{(n_2 + n_1)^2} = \frac{n_2^2 - 2n_1 n_2 + n_1^2 + 4n_1 n_2}{(n_1 + n_2)^2}$$

$$= \frac{(n_1 + n_2)^2}{(n_1 + n_2)^2} = 1$$

Evanescence *Find the e-folding distance, z_0 for the evanescent transmitted wave of TIR. Make a sketch of the moving "truncated plane wave." What about the $\vec{B}$ field for this situation?*

$$\vec{E}_t = \vec{E}_{t_0} \exp(i(\omega t - \vec{k}_2 \cdot \vec{r})) = \vec{E}_{t_0} \exp(i(\omega t - k_2 y \sin\theta_2 - i k_2 z \cos\theta_2))$$

Making the substitutions:

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2}, \ \sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1, \ \frac{n_2}{n_1} = \sin\theta_{crit}$$

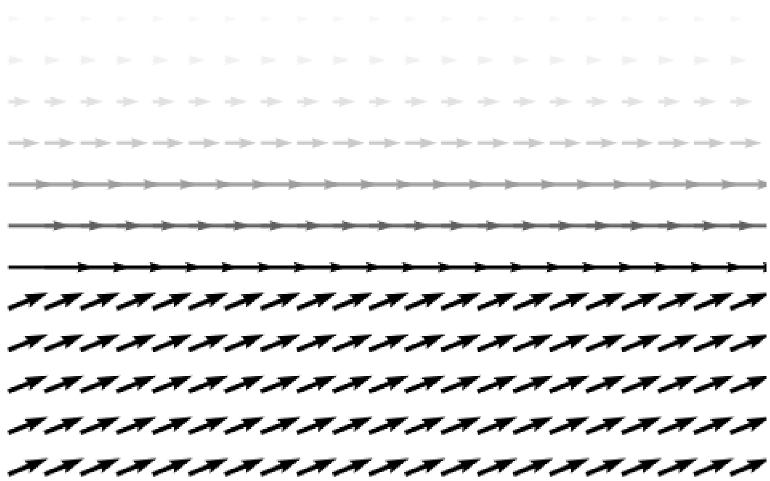
Then:

$$\vec{E}_t = \vec{E}_{t_0} \exp\left(-k_2 z \sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{crit}}\right)^2 - 1}\right) \exp\left(i\left(\omega t - k_2 y \frac{n_2}{n_1} \sin\theta_1\right)\right)$$

The first exponential is real and denotes the magnitude (and thus the damping). Since the exponent is $-z/z_0$, then

$$z_0 = \frac{1}{k_2\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{crit}}\right)^2 - 1}}$$

Figure 1: Vectors indicate the direction of $\vec{k}$, scaled to the relative magnitude E, near an interface. Opacity is is proportional to length (to make seeing the exponential decay of E easier).



Since this evanescent wave is traveling completely in the y direction, this wave is traveling parallel to the interface. Thus $\vec{E}$ and $\vec{B}$ of the evanescent wave both lie on planes perpendicular to the surface and perpendicular to $\hat{k}$. Since $E_0 \propto B_0$, the $\vec{B}$ field also diminishes exponentially w.r.t. distance from the interface like the $\vec{E}$ field.

Don't drink it? Given a material in which $\vec{M} = s\dot{\vec{P}}$:

(a) *Show that* $E \cdot \dot{E} = 0$.

$$\vec{M} = \chi_m \vec{H} = s\dot{\vec{P}} = s\frac{\partial}{\partial t}(\epsilon_0 \chi_e \vec{E}) = s\epsilon_0\chi_e\dot{\vec{E}}$$

$$\dot{\vec{E}} = c\vec{H},\ c = \frac{\chi_m}{s\epsilon_0\chi_e}$$

$$\vec{E}\cdot\dot{\vec{E}} = c(\vec{E}\cdot\dot{\vec{H}}) = c(0) = 0$$

(b) *Give a physical explanation for this result.*

This means that the vector $\vec{E}$ is rotating only (i.e., not changing magnitude).

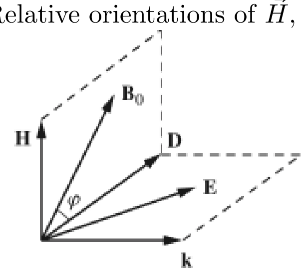
Figure 2: $\vec{E} \perp \dot{\vec{E}}$

Big magnets and powerful lasers! *The following constituent equations are given for the non-simple optical material, Leelanium:*

$$\vec{B} = \mu\vec{H},\ \vec{D} = \epsilon\vec{E} - i\gamma\vec{B}\times\vec{E}$$

Treat the wave as a plane wave with wave vector $\vec{k}$, where ϕ is the angle between the field of the magnet and the $\vec{D}$ field of the plane wave.

Figure 3: Relative orientations of $\vec{H}$, $\vec{B}_0$, $\vec{D}$, $\vec{E}$, $\hat{k}$



(a) *You have a very strong $7.5\,W$ laser and a strong magnetic field $\vec{B}_0$ of $1\,T$. Find the magnitude of the laser's $\vec{B}$ field, assuming the beam has a diameter of $2mm$. How large is B_0/B?*

Since $\vec{S}$ is an energy flux (energy per unit area per unit time, or power per unit area):

$$\langle S \rangle = \frac{P}{A} = \frac{7.5\text{W}}{\pi(1 \times 10^{-3}\text{m})^2} = 2.39 \times 10^6 \text{ W/m}^2$$

But also

$$\langle S \rangle = \langle u \rangle c = \frac{1}{2\mu_0} B^2 c$$

Thus

$$B = \sqrt{\frac{2\mu_0 \langle S \rangle}{c}} = \sqrt{\frac{2(4\pi \times 10^{-7}\text{H/m})(2.39 \times 10^6 \text{W/m}^2)}{3.00 \times 10^8 \text{m/s}}}$$

$$= 1.42 \times 10^{-4}\text{T}$$

Thus

$$\frac{B_0}{B} = \frac{1\text{T}}{1.42 \times 10^{-4}\text{T}} = 7.07 \times 10^4$$

Thus it is safe to assume $B_0 \gg B$.

(b) *Solve for the vector $\vec{H}$ in terms of $\vec{k}$ and $\vec{E}$. Then find the Poynting vector. Simplify and express the wave-vector as $\vec{k} = k\hat{k}$.*

Finding $\vec{H}$:

$$\vec{B} = \mu \vec{H}$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\mu \vec{H})$$

$$i\vec{k} \times \vec{E} = i\omega\mu\vec{H}$$

$$\vec{H} = \frac{1}{\mu\omega}\vec{k} \times \vec{E}$$

Finding $\vec{S}$:

$$\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \left(\frac{1}{\mu\omega}\vec{k} \times \vec{E}\right) = \frac{1}{\mu\omega}\left(\vec{k}(\vec{E} \cdot \vec{E}) - (\vec{k} \cdot \vec{E})\vec{E}\right)$$

$$= \frac{k}{\mu\omega}\left(E^2\hat{k} - (\hat{k} \cdot \vec{E})\vec{E}\right)$$

(c) *Use $\nabla \cdot \vec{D}$ to show that $\hat{k} \cdot \vec{E} = -i\gamma/\epsilon^2 B_0 D \sin\phi$. Assume (i) $B_0 \gg B$, and (ii) $\vec{E} \approx \vec{D}/\epsilon$.*

$$\nabla \cdot \vec{D} = i\vec{k} \cdot \vec{D} = 0 \Rightarrow \hat{k} \cdot \vec{D} = 0$$

$$\hat{k} \cdot (\epsilon\vec{E} - i\gamma\vec{B} \times \vec{E}) = \epsilon\hat{k} \cdot \vec{E} - i\gamma\hat{k} \cdot (\vec{B} \times \vec{E}) = 0$$

Using the approximations, we can replace $\vec{E} = \vec{D}/\epsilon$, and since almost all of the $\vec{B}$ field comes from $\vec{B_0}$, replace $\vec{B} = \vec{B_0}$.1

$$\hat{k} \cdot \vec{E} = \frac{i\gamma}{\epsilon} \hat{k} \cdot (\vec{B} \times \vec{E}) = \frac{i\gamma}{\epsilon^2} \hat{k} \cdot (\vec{B_0} \times \vec{D})$$

From (Figure 3), we can tell that $\vec{B_0} \times \vec{D}$ is in the direction $-\hat{k}$. Thus this further simplifies to

$$\hat{k} \cdot \vec{E} = \frac{i\gamma}{\epsilon^2} \hat{k} \cdot (B_0 D \sin\phi(-\hat{k})) = \frac{i\gamma}{\epsilon^2} B_0 D \sin\phi(\hat{k} \cdot (-\hat{k})) = -\frac{i\gamma}{\epsilon^2} B_0 D \sin\phi$$

(d) *Use (c) to simplify the expression for $\vec{S}$ from (b). Explain why this result shows the laser bends away from the direction it would travel in simple material. At what angle ϕ is the bending at a maximum?*

$$\vec{S} = \frac{k}{\mu\omega} \left(E^2 \hat{k} - \left(-\frac{i\gamma}{\epsilon^2} B_0 D \sin\phi \right) \vec{E} \right) = \frac{k}{\mu\omega} \left(E^2 \hat{k} + \frac{i\gamma}{\epsilon^2} B_0 D \sin\phi \vec{E} \right)$$

In simple material (or without a magnetic field), $\vec{S}$ should be only in the direction of $\hat{k}$ (in the direction of wave propagation). Here, we see some component of the Poynting vector in the direction of $\vec{E}$, which is almost perpendicular to $\vec{k}$.

The bending of $\vec{S}$ is greatest when $\sin\phi = 1$, i.e., when $\phi = \pi/2$, i.e., when the external (strong) $\vec{B_0}$ field is perpendicular to the $\vec{E}$ of the wave.