PH214C – PSET3

Jonathan Lam

February 26, 2020

Tale of two sigmas The Thomson scattering cross-section is given by

$$\sigma_{th} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2$$

Plugging in values:

$$q = \pm 1.602 \times 10^{-19}\,\mathrm{C}$$
$$\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F/m}$$
$$c = 2.998 \times 10^{8}\,\mathrm{m/s}$$
$$m_e = 9.109 \times 10^{-31}\,\mathrm{kg},\ m_p = 1.673 \times 10^{-27}\,\mathrm{kg}$$

We get:

$$\sigma_e = 6.650 \times 10^{-29}\,\mathrm{m}^2,\ \sigma_p = 1.971 \times 10^{-35}\,\mathrm{m}^2$$

It makes sense that the cross-section of an electron's scattering area is larger than that of a proton, since the electron is free to move over a much larger area than the proton is; i.e., assuming an atom's center-of-mass is fixed, the volume (and therefore surface area) occupied by the nucleus is tiny compared to the amount of volume and surface area the electrons are allowed to roam, and therefore it makes sense that the electron scatters more radiation per the same overall area.

The "classical electron radius" is the term inside the square, i.e., $\frac{q^2}{4\pi\epsilon_0 mc^2}$:

$$r_{0_e} = 2.817 \times 10^{-15}\,\mathrm{m}$$

N_2 **blues** N_2 characteristic frequency ("transition") has $\lambda \approx 75\,\mathrm{nm}$ (ultraviolet). Number density per unit surface area "footprint" on Earth's surface of N_2 is $n = 1.68 \times 10^{25}\,\mathrm{cm}^{-2}$.

a) Blue sunlight has $\lambda \approx 450\,\mathrm{nm}$. The transmitted energy flux through the atmosphere

$$\langle S\rangle_{tr} = \langle S_0\rangle e^{-N\sigma z}$$

where S_0 is the initial unscattered energy flux, N is the scatterer number density (per unit volume), σ is the scattering cross-section, and z is the distance the light has to travel through. In this case, $Nz = n$ is the number of scatterers per cross-sectional area. Thus the percentage of light scattered is

$$\% \text{ light scattered} = \frac{\langle S_0 \rangle - \langle S_0 \rangle e^{-n\sigma}}{\langle S_0 \rangle} = 1 - e^{-n\sigma}$$

where σ is the Rayleigh scattering cross-section.

$$\omega = \frac{2\pi c}{\lambda} = 4.186 \times 10^{15} \text{rad/s}; \ \omega_0 = \frac{2\pi c}{\lambda_0} = 2.512 \times 10^{16} \text{rad/s}$$

$$\sigma_{ray} = \left(\frac{\omega^2}{\omega^2 - \omega_0^2} \right)^2 \sigma_{th}$$

$$= \left(\frac{(4.186 \times 10^{15} \text{ rad/s})^2}{(4.186 \times 10^{15} \text{ rad/s})^2 - (2.512 \times 10^{16} \text{ rad/s})^2} \right)^2 (6.650 \times 10^{-29} \text{ m}^2)$$

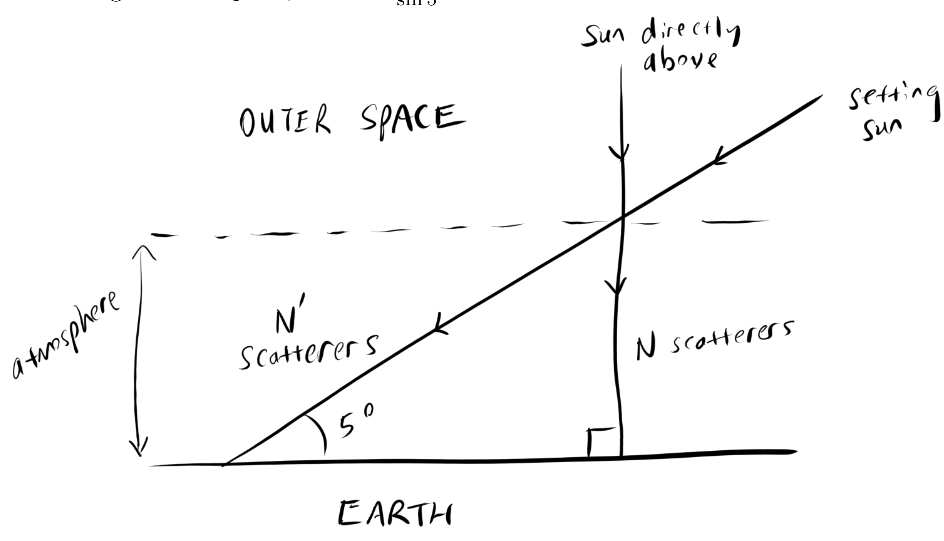
$$= 5.425 \times 10^{-32} \text{ m}^2$$

Thus

$$\% \text{ light scattered} = 1 - \exp(-(1.68 \times 10^{29} \text{ m}^{-2})(5.425 \times 10^{-32} \text{ m}^2))$$

$$= 0.907\%$$

b) The path taken by a wave of light from the Sun when the angle of elevation is $5°$ is much longer than that when it is shining directly overhead. Assuming that the Earth is a planar slab and the atmosphere is a constant height above the surface of the Earth, the ratio of the vertical path vs. the path at $5°$ elevation is $\sin 5°$. The number of electrons that scatter the blue light should be roughly proportional to the length of the path, so $n' = \frac{n}{\sin 5°}$.



=

Thus the relative amount of light scattered is

$$\% \text{ light scattered} = 1 - e^{-n'\sigma} = 1 - \exp\left(-\frac{n\sigma}{\sin(5^\circ)}\right) = 9.93\%$$

v and c The Maxwell equations for materials:

$$\nabla \cdot \vec{D} = 0,\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},\ \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\vec{D} = \epsilon \vec{E},\ \vec{B} = \mu \vec{H}$$

a) Solving the wave equation for $\vec{E}$ and $\vec{B}$:

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\nabla(0) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H})$$

$$\nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\frac{\partial \vec{D}}{\partial t} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

This is the wave equation for $\vec{E}$ with $v = \frac{1}{\sqrt{\mu\epsilon}}$. Similarly, for $\vec{B}$:

$$\nabla \times \vec{H} = \frac{1}{\mu}(\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}) = \nabla \times \left(\frac{\partial \vec{D}}{\partial t}\right)$$

$$\frac{1}{\mu}(\nabla(0) - \nabla^2 \vec{B}) = \epsilon \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$-\frac{1}{\mu}\nabla^2 \vec{B} = \epsilon \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

This is the wave equation for $\vec{B}$ with the same v. Since $\vec{B}$ and $\vec{E}$ are scaled versions of $\vec{D}$ and $\vec{H}$ and the wave equation is linear, these fields also are solutions to the wave equation.

b) $\kappa_e = \epsilon/\epsilon_0 = 1 + \chi_e$ (result from lecture). Let $\mu \approx \mu_0$. Then

$$n = \frac{c}{v} = \frac{(\mu_0\epsilon_0)^{-1/2}}{(\mu_0\epsilon)^{-1/2}} = \sqrt{\frac{\mu_0\epsilon}{\mu_0\epsilon_0}} = \sqrt{\kappa_e}$$

Dielectric raindrops and rods The subscript $_1$ indicates entities just outside the droplet, and $_2$ indicates those inside the raindrop. The $\vec{E}$-field within the raindrop is uniform and horizontal. Boundary conditions imposed by material Maxwell's equations:

$$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1 = 0), \;\; \hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$$

Normal component of $\vec{E}_1$:

$$\vec{E}_1^{\perp} = \frac{1}{\epsilon_1}\vec{D}_1^{\perp} = \frac{1}{\epsilon_1}\vec{D}_2^{\perp} = \frac{1}{\epsilon_1}\vec{D}_2 \cos\theta = \frac{\epsilon_2}{\epsilon_1}\vec{E}_2 \cdot \hat{r}$$

Tangential component of $\vec{E}_1$:

$$\vec{E}_1^{||} = \vec{E}_2^{||} = \vec{E}_2 \times \hat{r}$$

Let $\epsilon_2/\epsilon_1 = 2$. Thus, the vector $\vec{E}_1$ in $(\hat{r}, \hat{\theta})$ coordinates is

$$\left(2\vec{E}_2 \cdot \hat{r}, \vec{E}_2 \times \hat{r}\right) = (2E_2 \cos\theta, E_2 \sin\theta)$$

In Cartesian coordinates $(\hat{x}, \hat{y})$, this is:

$$\left(2E_2 \cos^2\theta + E_2 \sin\theta \cos\left(\theta - \frac{\pi}{2}\right), 2E_2 \cos\theta \sin\theta + E_2 \sin\theta \sin\left(\theta - \frac{\pi}{2}\right)\right)$$

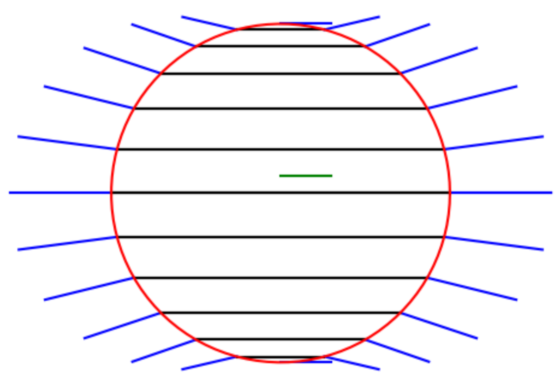


Figure 2. Droplet and E-fields. Black lines indicate the horizontal orientation of the $\vec{E}$ field. The blue vectors indicate the $\vec{E}$ field directly outside the droplet. The green vector indicates the magnitude of the (uniform) $\vec{E}$ within the droplet. The relative magnitude of the blue vectors and the green vectors are drawn to scale, with $\epsilon_2/\epsilon_1 = 2$. (The green vector and blue vectors are all pointing right, but arrowheads are not shown.)