PH214: Optics and Modern Physics

Jonathan Lam

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Commentary on Prof. Yecko's and Prof. Debroy's wonderful lecture notes. No guarantees on correctness or completeness. Not meant to be standalone, but to supplement the lecture notes and their diagrams.

Contents

1 Maxwell's equations

1.1 General form

Illustrated previously in E&M. In order: Gauss's flux laws (electric and magnetic); Faraday's induction law; Ampere's current law. Shown in integral and differential forms.

$$\oiint_{\partial V} \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon} \iiint_V \rho \, dV \equiv \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \tag{1}$$

$$\oiint_{\partial V} \vec{B} \cdot d\vec{s} = 0 \equiv \nabla \cdot \vec{B} = 0 \tag{2}$$

$$\oint_C \vec{E} \cdot d\vec{l} = -\iint_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \equiv \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\oint_C \vec{B} \cdot d\vec{l} = \mu \iint_A \left(\vec{j} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{s} \equiv \nabla \times \vec{B} = \mu \left(\vec{j} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) \tag{4}$$

where ε is permittivity, μ is the permeability, and $\vec{j}$ is current density.

1.2 In a vacuum

In a vacuum, there is no charge density, and thus no current density. Thus Maxwell's equations in a vacuum (in differential form) simplify to

$$\nabla \cdot \vec{E} = 0 \tag{5}$$

$$\nabla \cdot \vec{B} = 0 \tag{6}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7}$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{8}$$

where ε_0 is the permittivity of free space and μ_0 is the permeability of free space.

1.3 Review of differential operators

1.3.1 The del operator ∇

Note that the nabla/del operator in the below forms are principally defined for use with Cartesian coordinates. In polar (cylindrical or spherical) coordinates, these operations have to be converted from these Cartesian equivalents. It can be treated as a "vector":

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{9}$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \tag{10}$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{11}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{12}$$

The Laplacian is a second-order differential operator, and can be treated as a "scalar" operator obtained from the dot product of nabla with itself:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{13}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{14}$$

$$\nabla^2 \vec{A} = \left(\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z\right) \tag{15}$$

1.4 Other useful identities

$$\vec{A} \cdot (\vec{A} \times \vec{B}) = 0 \tag{16}$$

$$\vec{A} \cdot (\vec{A} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} \tag{17}$$

$$\nabla \times \nabla f = 0 \text{ (curl of gradient field is zero)} \tag{18}$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \text{ (divergence of curl field is zero)} \tag{19}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \tag{20}$$

$$\nabla \times (\vec{A} \cdot \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} \tag{21}$$

1.5 Proving the wave equation from Maxwell's equations

The canonical form for a wave (what we desire to achieve):

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2} \tag{22}$$

Remembering the side of v makes sense since it is time over distance, canceling out the units. Assuming a vacuum, we start with (Eq. 7) and apply identity (Eq. 20):

$$\nabla \times \left(\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{23}$$

This is in the form of (Eq. 22) in the 3D case, with $c^2 = \frac{1}{\mu_0 \varepsilon_0}$. The same can be shown for $\vec{B}$.

Note that since we know that sinusoids are solutions to the wave equation, then any periodic $\vec{E}$ function is a solution for this $\vec{E}$ wave equation (since it can be decomposed into a linear combination of sinusoids using Fourier analysis); this would then force constraints on $\vec{B}$ (and vice versa for arbitrary $\vec{B}$ waves). If there is a pair of matching $\vec{E}$ and $\vec{B}$ oscillations in space that match both wave equations simultaneously and satisfies the Maxwell equations, then it is an EM wave.

1.6 Polarization of an EM wave

For now, assume that $\vec{E}$ (and $\vec{B}$) waves are transverse (i.e., a transverse electromagnetic wave (TEM); not proven yet; see (Eq: 35)); i.e., $\vec{E} \cdot \hat{k} = 0$. Assume the wave is traveling in the z direction. Then

$$\vec{E} = E_x(z,t)\hat{\imath} + E_y(z,t)\hat{\jmath} \tag{24}$$

Since $E_x\hat{\imath}$ and $E_y\hat{\jmath}$ are themselves waves, then E_x and E_y both fit the form:

$$f(z,t) = A\cos(k(z-ct)+\delta) \tag{25}$$

If $\delta_x = \delta_y$, then $\vec{E}$ is linearly polarized. If $\delta_x \neq \delta_y$, then $\vec{E}$ is elliptically polarized. If $\delta_x \neq \delta_y$ and $E_x = E_y$, then $\vec{E}$ is circularly polarized. We can also write $\vec{E}$ as:

$$\vec{E} = E\cos\theta\hat{\imath} + E\sin\theta\hat{\jmath} \tag{26}$$

and we call θ the polarization angle.

Polarization comes from the direction of acceleration of a charge, which will be seen in the scattering section.

1.7 Plane waves

We can express a simple (real) cosine wave as its phasor equivalent:

$$\vec{E} = E_0\cos(k(z-ct)+\delta)\hat{n} = \Re\left(E_0 e^{i(k(z-ct))}\right)\hat{n} \tag{27}$$

where E_0 is a real constant in the cosine form, and a complex constant (including the phase) in the phasor form, and $\hat{n}$ is the unit vector in the direction of the polarization. k is the wave number, and c is the speed of light.

Alternatively, we can express the phase as $i(kz-\omega t)$, recognizing that $\omega = kc$. (Also, $c = \omega/k$, and $k = \omega/c$.)

More generally, if a wave is propagating in direction $\hat{k}$ with wave number k, we define the wave vector $\vec{k}$ and express the wave as

$$\vec{E} = \Re\left(E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)\hat{n} \tag{28}$$

1.7.1 Plane wave derivatives

Let $\vec{E}$ be a plane wave. Then:

$$\frac{\partial \vec{E}}{\partial t} = -i\omega\vec{E} \tag{29}$$

$$\nabla\cdot\vec{E} = i\vec{k}\cdot\vec{E} \tag{30}$$

$$\nabla\times\vec{E} = i\vec{k}\times\vec{E} \tag{31}$$

In other words, we have correspondences between the differential operators and multiplication (for plane waves). In particular:

$$\frac{\partial}{\partial t} \leftrightarrow -i\omega \tag{32}$$

$$\nabla \leftrightarrow i\vec{k} \tag{33}$$

$$\nabla^2 = \nabla\cdot\nabla \leftrightarrow i\vec{k}\cdot i\vec{k} = -k^2 \tag{34}$$

Thus, to prove that plane waves are transverse:

$$\nabla\cdot\vec{E} = i\vec{k}\cdot\vec{E} = 0 \Rightarrow \vec{k}\perp\vec{E} \tag{35}$$

(The same result is true for $\vec{B}$.) To find the relative magnitudes of $\vec{E}$ and $\vec{B}$:

$$\nabla\times\vec{E} = i\vec{k}\cdot\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B} \Rightarrow \vec{k}\times\vec{E} = \omega\vec{B} \tag{36}$$

$$\nabla\times\vec{B} = i\vec{k}\cdot\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = -\frac{i\omega}{c^2}\vec{E} \Rightarrow \vec{k}\times\vec{B} = -\frac{\omega}{c^2}\vec{E} \tag{37}$$

Note that (Eq. 36) shows that $\vec{B}$ is also transverse since it results from a cross product with $\vec{E}$. The negative sign arises from a right-handed coordinate system. Since $\vec{k}$, $\vec{B}$, $\vec{E}$ are mutually orthogonal, the cross products become the products of the magnitudes, i.e.:

$$kE = \omega B \Rightarrow B = \frac{k}{\omega}E = \frac{1}{c}E \tag{38}$$

Note that this factor of $\frac{1}{c}$ is meaningless; it is more or less a consequence of our unit systems of $\vec{E}$ and $\vec{B}$ fields.

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2 Poynting vector

2.1 Energy density of E and B fields

Denote energy density with u. These equations were derived in E&M.

$$u_E = \frac{1}{2}\varepsilon_0 \vec{E} \cdot \vec{E} = \frac{1}{2}\varepsilon_0 |\vec{E}|^2 \tag{39}$$

$$u_B = \frac{1}{2\mu_0}\vec{B} \cdot \vec{B} = \frac{1}{2\mu_0}|\vec{B}|^2 \tag{40}$$

For an E&M wave, at any point in space and moment in time, $u_E = u_B$, so total energy density is $u = u_E + u_B = 2u_E = 2u_B$.

2.2 Poynting vector

The Poynting vector points in the direction of (outward) energy flux. It can be thought of as the energy density multiplied by a velocity.

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{41}$$

To derive this, calculate the rate of change in energy density.

$$\frac{du}{dt} = \frac{\partial}{\partial t}\left(\frac{1}{\mu_0}\vec{B} \cdot \vec{B}\right) = \frac{1}{\mu_0}\frac{\partial \vec{B}}{\partial t} \cdot \vec{B} = \frac{1}{\mu_0}(-\nabla \times \vec{E}) \cdot \vec{B} = -\nabla \cdot \left(\frac{1}{\mu_0}\vec{E} \times \vec{B}\right) = -\nabla \cdot \vec{S} \tag{42}$$

This makes sense from a conservation-of-energy perspective (this is called a continuity statement):

$$\frac{du}{dt} + \nabla \cdot \vec{S} = 0 \tag{43}$$

This statement can be read as: the rate of change of (internal) energy is equal to the energy flux (or divergence, or rate of loss of energy).
Time-averaging the Poynting vector:

$$\begin{aligned}\langle \vec{S} \rangle = \frac{1}{T}\int_0^T \frac{1}{\mu_0}\vec{E} \times \vec{B}\, dt = \frac{1}{T\mu_0}E_0 B_0 \int_0^T sin^2[\phi(t)]\, dt = \frac{1}{2\mu_0}E_0 B_0 \\ = \frac{1}{2}c\varepsilon_0 E_0^2 \hat{k}\end{aligned} \tag{44}$$

where $\hat{k}$ is the unit vector in the direction of the wave's propagation. Since $\langle u \rangle = \frac{1}{2}\varepsilon_0 E_0^2$, we obtain the "Suck" equation:

$$\langle \vec{S} \rangle = \langle u \rangle c \hat{k} \tag{45}$$

This is as we interpreted at the beginning of this section: the energy flux is an energy density being carried at the wave's propagation speed. The units for this are W/m^2; the rate of energy flux penetrating some patch of closed loop in space.

Lastly, we define intensity as the magnitude of $\langle \bar{S} \rangle$, i.e.,

$$I = |\langle \bar{S} \rangle| = \langle u \rangle c \tag{46}$$

2.3 Energy density as pressure

It makes sense that packing more energy in a unit volume may seem to give it more "pressure." Working dimensionally, this also works:

$$P = \frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{N} \cdot \mathrm{m}}{\mathrm{m}^2 \cdot \mathrm{m}} = \frac{\mathrm{J}}{\mathrm{V}} = u \tag{47}$$

This "light pressure" is theoretically able to be able to exert forces on objects, but its applicability is only theoretical right now (see "solar sails").

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3 Radiation and scattering

After receiving an impulse from an EM wave, an electron radiates energy. We only study Thomson (free electron) and Rayleigh (electron bound to an atom) scattering, both of which are elastic scattering models.

3.1 Radiation

First, we talk about radiation due to an atom due to some acceleration. When radiation comes into contact with matter, it may be reflected, refracted, absorbed, or scattered. In this section we will discuss scattering models only.

3.1.1 Setup and fundamental results

An electron is at rest at the origin. It is briefly accelerated (wlog., in the positive x-direction) for a short interval of time τ, after which it moves at constant velocity $v \ll c$. We create a model to examine its $\vec{E}$ field at time $T \gg \tau$. Since EM waves travel at the speed of light, the EM wave updates in "rings" propagating outward from the particle at c.

The first boundary is at radius $R_0 = cT$; outside of this circle, an observer doesn't know the particle has moved. Since the acceleration stops after τ, the second important boundary is at $R_I = c(T - \tau)$; inside this circle, an observer sees the charge moving at a uniform speed. We make the approximation that these two circles are centered at the origin, since $\tau \ll T$.

Now, picture an $\vec{E}$ wave leaving the electron at angle θ. It travels in a straight line until the first boundary. Outside of the second boundary, however, an observer doesn't know the electron has moved from the origin yet, however, so the ray outside of the second boundary would be a straight line towards the origin, also roughly at angle θ. Thus, in the region between these two boundaries (where the information about the electron's acceleration is currently reaching), there is a "kink" in the electric field waves; we approximate this with a straight line. We can also approximate the curvature of the two circles to be small in this region, and assume them to be straight lines in the tangential direction.

Now we have a "rectangle" with side lengths E_{kink_r} and E_{kink_θ} (since $\vec{E}_{kink}$ is the diagonal). From the geometry and approximations given (now, let $\vec{E} = \vec{E}_{kink}$):

$$E_\theta = vT \sin\theta$$

$$E_r = c\tau$$

$$\frac{E_\theta}{E_r} = \frac{vT \sin\theta}{c\tau}$$

The following substitutions are also appropriate given the setup:

$$R = cT$$

$$a = \frac{v}{\tau}$$

From Coulomb's/Gauss's law for a point charge, we know that the radial $\vec{E}$ field is also:

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2} \tag{48}$$

So we can rewrite E_θ as (using the appropriate substitutions):

$$E_\theta = \frac{qa\sin\theta}{4\pi\epsilon_0 c^2 R} \tag{49}$$

Note that, as R increases, $\vec{E}_\theta$ dominates over $\vec{E}_r$ ($E_\theta \propto R^{-1}$, not R^{-2} like radial $\vec{E}$). Thus, for reasonable distances, is practically a transverse wave with magnitude E_θ caused by an acceleration of the charge at the retarded time $-R/c$. This $\vec{E}_\theta$ is thus the $\vec{E}$ field of the radiated EM wave; we are not concerned with the radial component. Using (Eq. 38), we have:

$$B = \frac{qa\sin\theta}{4\pi\epsilon_0 c^3 R} \tag{50}$$

We can also find the Poynting vector associated with this EM wave. Using (Eq. 41), we can find the magnitude of the Poynting flux:

$$S = \frac{q^2 a^2 \sin^2\theta}{16\pi^2\epsilon_0 r^2 c^3} \tag{51}$$

To recap, these are the magnitudes of the $\vec{E}$, $\vec{B}$, and $\vec{S}$ fields for the EM wave radiated by a charge accelerated at some retarded time, where $\vec{E}$ and $\vec{B}$ are in the tangential directions and mutually perpendicular, and $\vec{S}$ is in the radial direction.

3.1.2 Geometry of the radiation

The $\vec{E}$, $\vec{B}$, and $\vec{S}$ fields are clearly anisotropic (are a function of angle). Looking at the $\vec{S}$ vector, we can see that the power radiated is proportional to $\sin^2\theta$, and thus is sort of a weird donut-y shape. Specifically, in the direction of acceleration, there is no radiation; in the perpendicular direction, there is the maximum ("beaming") with magnitude

$$S = \frac{q^2 a^2}{16\pi^2 \epsilon_0 r^2 c^3} \tag{52}$$

which can be clearly seen from (Eq. 51).

If the donut is sitting flat on a table, then the $\vec{E}$ vector is tangent to the surface and points up or down; the $\vec{B}$ is tangent to the surface and points around the donut. Since the orientation of the donut is dictated by the direction of acceleration, the directions of $\vec{E}$, $\vec{B}$, and $\vec{S}$ are determined by the direction of acceleration; this is where polarization comes from.

The region of no radiation will show up again later when encountering Brewster's angle.

3.1.3 Larmor power

To get the total radiated power (Larmor power), we integrate the Poynting vector flux through some closed surface containing the charge. For simplicity, choose a sphere (and integrate using cylindrical coordinates):

$$P = \int_0^\pi (S)(2\pi r)(r\sin\theta)d\theta = \frac{q^2 a^2}{8\pi\epsilon_0 c^3}\int_0^\pi \sin^3\theta\, d\theta = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \tag{53}$$

This is the power, or total energy loss by an accelerated charge (in Watts).

3.2 Thomson scattering

Thomson scattering is the model used for free electrons. Imagine an incident $\vec{E}$ wave approaching a free electron, and the wave is traveling in the $\hat{x}$ direction. Let:

$$E = E_x = E_0\cos(kz - \omega t) = E_0 \cos\omega t$$

Then:

$$a = a_x = \frac{qE_0}{m}\cos\omega t \tag{54}$$

Plugging into (Eq. 53), we get:

$$P(t) = \frac{q^2 a^2(t)}{6\pi\epsilon_0 c^3} = \frac{q^4 E_0^2}{6\pi\epsilon_0 m^2 c^3}\cos^2\omega t$$

Time averaging, we get:

$$\langle P \rangle_{scatt_{th}} = \frac{q^4 E_0^2}{12\pi\epsilon_0 m^2 c^3} \tag{55}$$

This is the Thomson scattering by a single electron due to an incident $\vec{E}$ wave with amplitude E_0. We may want to express this in terms of the incident power; from (Eq. 44), we can do precisely this (and some simplifying) to obtain:

$$\langle P \rangle_{scatt_{th}} = \frac{8\pi}{3} \left(\frac{q^2}{4\pi\epsilon_0 mc^2} \right)^2 \langle S \rangle_{inc} \tag{56}$$

Note that this equation works dimensionally: we have a power equal to some area multiplied by a power flux. In particular, the quantity inside the parens is a length and is called the classical electron radius, r_0:

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2} \tag{57}$$

and this area is called the Thomson scattering cross-section:

$$\sigma_{th} = \frac{8\pi}{3} r_0^2 \tag{58}$$

So we can abbreviate the Thomson scattering using this new constant:

$$\langle P \rangle_{scatt_{th}} = \sigma_{th} \langle S \rangle_{inc} \tag{59}$$

Note that the scattering coefficient is intrinsic of the material and not of the intensity of the light; that is, for a given scatterer, the scattered power is proportional to the incident power flux.

3.2.1 Application: transmission through the Sun's corona

We can model the Sun's corona as a bunch of free electrons and protons. The protons are much weaker scatterers (their σ_{th} value is much smaller due to their larger mass), so we'll ignore their contribution. The change in total power after passing through a length dz through a portion of the Sun's atmosphere is

$$dP = -A\, d\langle S \rangle = NA\, dz\, \sigma \langle S \rangle$$

where A is the cross-sectional area of interest (we can see this quickly cancels out), and N is the number density of scatterers. The right hand side of this equation comes directly from (Eq. 59) through $NA\, dz$ electrons. Rearranging, we get a simple ODE, to which the solution is

$$\langle S \rangle = \langle S_0 \rangle \exp(-N\sigma z) \tag{60}$$

For the solar corona, we get $\langle S \rangle \approx 0.99995 \langle S_0 \rangle$, so very little of the Sun's total radiative power is scattered by its corona (and that is why it is so hard to see).

3.3 Rayleigh scattering

Rayleigh scattering uses a mechanical "electron on a spring" model of the atom, and is more accurate for electrons in matter. For simplicity, we can treat it in the undamped case, and then move onto the damped case. We see that it is similar to Thompson scattering: the radiated power is also proportional to the incident power flux, except now the proportionality "constant" is frequency-dependent (and thus not really a "constant").

3.3.1 Setup and fundamental results

We treat the electron as an undamped spring with driving force $qE_0 \cos \omega t$, similar to the original setup for Thomson scattering. However, the acceleration is not as simple: we have the additional restoring force term. Luckily this is still just a linear ODE that we can solve normally just like we would for a spring mechanics problem:

$$m\frac{d^2x}{dt^2} + m\omega_0^2 x = qE_0 \cos \omega t \tag{61}$$

Solving:

$$x = \frac{qE_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \tag{62}$$

Solving for acceleration, we can plug this back into (Eq. 53) to get the Larmor power from Rayleigh scattering:

$$\frac{d^2x}{dt^2} = a = \frac{\omega^2}{\omega^2 - \omega_0^2} \frac{qE_0}{m} \cos \omega t \tag{63}$$

Compare this to (Eq. 54); the only difference is the frequency term. Since $S \propto a^2$, it can be seen from analogy to the derivation of Larmor power of Thomson scattering that the Larmor power for Rayleigh scattering is

$$\langle P \rangle_{scatt_{ray}} = \frac{8\pi}{3} \left(\frac{q^2}{4\pi\epsilon_0 mc^2} \right)^2 \left(\frac{\omega^2}{\omega^2 - \omega_0^2} \right)^2 \langle S \rangle_{inc} \tag{64}$$

If we express the scattering terms as a single coefficient, we get

$$\sigma_{ray} = \left(\frac{\omega^2}{\omega^2 - \omega_0^2} \right)^2 \sigma_{th} \tag{65}$$

and then we can express Rayleigh scattering as the simple equation just like (Eq. 59):

$$\langle P \rangle_{scatt_{ray}} = \sigma_{ray} \langle S \rangle_{inc} \tag{66}$$

3.3.2 Properties and results from frequency dependence

As ω_0 goes to zero, then the restoring force becomes negligible, and thus it should behave like a Thompson scattering. Indeed,

$$\lim_{\omega_0 \to 0} \sigma_{ray} = \sigma_{th} \tag{67}$$

On the other hand, for gases, $\omega_0 \gg \omega$, and we can make the approximation:

$$\langle P\rangle_{scatt_{ray}} \approx \left(\frac{\omega}{\omega_0}\right)^4 \sigma_{th}\langle S\rangle_{inc} \tag{68}$$

This fourth-power means that higher frequencies are scattered much more strongly than lower ones, and causes the phenomenon that blue light is scattered much more strongly by the atmosphere, hence causing it to appear blue.

3.3.3 Resonance and damping

The undamped model doesn't make sense, because it would imply extremely unstable conditions when resonance occurs. Thus it makes sense to introduce a damping term, making the original differential equation into

$$m\frac{d^2x}{dt^2} + m\beta\frac{dx}{dt} + m\omega_0^2 x = qE_0\cos\omega t \tag{69}$$

The general equation for this is:

$$\vec{x} = \frac{q}{m((\omega_0^2 - \omega^2) + i\gamma\omega)}\vec{E} \tag{70}$$

This equation shows the movement of an electron relative to an atom; in other words, it shows the displacement of charges, or the dipole moment. In a static $\vec{E}$ field, then the driving frequency $\omega = 0$, and there is an average dipole moment in the same direction with magnitude:

$$\langle\vec{p}\rangle = q\langle\vec{d}\rangle = q\langle\vec{x}\rangle = \frac{q^2}{m\omega_0^2}\langle\vec{E}_{ext}\rangle \tag{71}$$

but we will not directly reference this solution to the undamped electron-on-a-spring equation until (Sec. 7.1) on absorption, which will deal with electrons responding to an incident $\vec{E}$ wave.

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4 Dipoles and Potentials

We already know from integrating the $\vec{E}$ field that the electric potential is a scalar field:

$$\phi_E = -\nabla\phi_E \tag{72}$$

We will obtain the same for the "magnetic field strength", the $\vec{H}$ (not yet revealed).

$$\phi_H = -\nabla\phi_M \tag{73}$$

For a point charge, ϕ_E is already known:

$$\phi_E = \frac{q}{4\pi\epsilon_0}\frac{1}{r} \tag{74}$$

4.1 Dipoles and beyond

Electric dipoles are of interest because they are created when molecules distort in an electric field. I.e., displacing a charge is equivalent to adding a dipole. Define $\vec{d}$ as the vector from the vector from the negative charge to the positive charge in a dipole. Also, define $\vec{p}$, the dipole moment, to be

$$\vec{p} = q\vec{d} \tag{75}$$

There are electric also monopoles, quadrupoles, octupoles, etc. They can be in either a 3-D general form or a linear (axi-symmetric) form, but both have the same potential magnitudes. For a monopole, the potential is proportional to r^{-1}; for a dipole, it is r^{-2}; and so on. We deal primarily with the axi-symmetric forms.

4.2 Potential of a dipole

We already know from E&M that to calculate the voltage at any point in space, we add up the contributions from all of the charges (or poles). For a dipole, in general, if we let $\vec{r}$ be the positional vector of the point of interest w.r.t. to the center of the dipole, and define:

$$\vec{r}_+ = \vec{r} - \frac{1}{2}\vec{d},\ \vec{r}_- = \vec{r} + \frac{1}{2}\vec{d}$$

(i.e., r_+ is the distance between the point of interest to the positive charge, and likewise for r_- and the negative charge) then:

$$\phi_{dip} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{76}$$

From the definition of $\vec{r}_+$ and $\vec{r}_-$, we can represent r_+^{-1} and r_-^{-1} as $|\vec{r} - \vec{\delta}|^{-1}$, where $\vec{\delta} = \pm\frac{1}{2}\vec{d}$. We can rewrite this in three ways:

$$\frac{1}{|\vec{r} - \vec{\delta}|} = \begin{cases} \frac{1}{r} - \vec{\delta}\cdot\nabla\left(\frac{1}{r}\right) + \cdots \\ \frac{1}{r}\left(1 - 2\left(\hat{\delta}\cdot\hat{r}\frac{\delta}{r}\right) + \frac{\delta^2}{r^2}\right)^{-1/2} \\ \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{\delta}{r}\right)^l P_l(\hat{\delta}\cdot\hat{r}) \end{cases} \tag{77}$$

where P_l is the lth Legendre polynomial. Note:

1. The former is the 3-D Taylor expansion.
2. The second is the law of cosines in the denominator.
3. The latter is a series expansion of the second. (Note that Legendre polynomials result from performing Gram Schmidt orthonormalization on the polynomial standard ordered basis (i.e., an orthonormal set created from

$\{1, x, x^2, x^3, \dots\}$), so you can write any function as an expansion (linear combination) of the Legendre polynomials by using the inner product of the function with a Legendre polynomial as the coefficicent.)

Using the first two terms of the first representation, we can see that the potential of a dipole is:

$$\begin{aligned}\phi_E &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-\frac{1}{2}\vec{d}|}-\frac{1}{|\vec{r}+\frac{1}{2}\vec{d}|}\right)\\ &\approx \frac{q}{4\pi\epsilon_0}\left(\left[\frac{1}{r}+\frac{1}{2}\vec{d}\cdot\nabla\left(\frac{1}{r}\right)\right]-\left[\frac{1}{r}-\frac{1}{2}\vec{d}\cdot\nabla\left(\frac{1}{r}\right)\right]\right)\\ &= \frac{q}{4\pi\epsilon_0}\left(\vec{d}\cdot\nabla\left(\frac{1}{r}\right)\right)=\frac{1}{4\pi\epsilon_0 r^2}\vec{p}\cdot\hat{r} \qquad (78)\end{aligned}$$

This is the single-term approximation for a dipole moment, and it is clearly proportional to r^{-2}. However, there are more terms in the expansion, but this approximation is considered good enough. This expression is not an approximation for a "point dipole"; i.e., if $d \to 0$ but $\vec{p}$ is finite (i.e., then $q \to \infty$).

4.3 Potentials inside/outside a distribution of charges

With the third representation from (Eq. 77), we may express potentials within a net-neutral distribution of charges. For a point outside of the charge distribution, then the potential can be expressed as:

$$\phi_{out} = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{A_l}{r^{l+1}}P_l(\cos\theta) \qquad (79)$$

and inside the charge distribution, the potential can be expressed as:

$$\phi_{in} = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}B_l r^l P_l(\cos\theta) \qquad (80)$$

(Not sure how to derive these specifically. You can see the general pattern, however, that outside the material, we involve positive coefficients of r^{-1}; inside the material we have nonpositive coefficients of r^{-1}.) Generally, we have to solve for the coefficients $\{A_l\}$ and $\{B_l\}$ given some other constraint.

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5 EM Fields in dielectric materials

The PDF at `http://www.phys.nthu.edu.tw/~thschang/notes/EM04.pdf` was very helpful for this topic.

5.1 The polarization and displacement fields

5.1.1 The polarization field

If we apply an $\vec{E}$ field on a dielectric material, then dipoles form. The net of these molecular dipoles forms the polarization field, $\vec{P}$:

$$\vec{P} = N\langle\vec{p}\rangle \tag{81}$$

where N is the number density of the dielectric particles. This can be thought of as a "dipole moment density." This is also proportional to the applied $\vec{E}$ field:

$$\vec{P} = \epsilon_0\chi_E\vec{E} \tag{82}$$

where χ_E is the dielectric susceptibility (can be thought of as how willing a substance is to polarizing; free space doesn't polarize, so $\chi_{E_0} = 0$). For this course, we concern ourselves with linear material; i.e., $\vec{D}$ is directly proportional to the applied $\vec{E}$, or χ_E is a constant proportionality factor. (Real materials are not always so simple.)

5.1.2 The displacement fields

The electric and polarization fields combine to form the displacement field $\vec{D}$, which is like the net dipole moment density:

$$\vec{D} = \epsilon_0\vec{E} + \epsilon_0\chi_E\vec{E} = \epsilon_0(1+\chi_E)\vec{E} \tag{83}$$

We define the relative permittivity κ as follows:

$$\kappa := 1 + \chi_E \tag{84}$$

and electric permittivity (of material) to be:

$$\epsilon = \kappa\epsilon_0 \tag{85}$$

(This gives the name "relative permittivity" meaning, as $\kappa = \epsilon/\epsilon_0$.) Thus, with these new definitions,

$$\vec{D} = \epsilon\vec{E} \tag{86}$$

From an intuitive perspective, this means that the displacement field (think of it as some sort of net electric field) is the applied electric field along with some induced amount of electric field, the amount that is induced is dependent on the material.

Taken directly from the notes: "Another way to think of the $\vec{P}$ field is in terms of 'bound charge density,' ρ_b, where:

$$-\nabla\cdot\vec{P} = \rho_b \tag{87}$$

For example, in [a dipole distribution in which all $\vec{p}$ vectors point outwards from a single point], $\nabla\cdot\vec{P} \neq 0$." In other words, the negative of the divergence of the polarization field is the "bound charge density," the net charge density due to dipolar (thus bound) electrons. In this particular dipole distribution, there is a net divergence, and thus $\rho_{bound} \neq 0$; intuitively, this makes sense since if all of the dipole moments are pointing outwards, then there is a positive polarization field divergence; also, the dipoles are slightly stretched, and the net interior charge due to the dipoles is slightly negative (since the surface/boundary charge is slightly more positive, and the total charge must be zero).

5.2 The $\vec{H}$ and magnetization fields

5.2.1 The $\vec{H}$ field

We define the field $\vec{H}$ (sometimes called the magnetic field strength field) as a more primitive "magnetic" field, one that doesn't rely on matter and is the magnetic analog to the $\vec{E}$ field. This is because the $\vec{B}$ field that is often used is actually more of an empirical measure traditionally used because we typically see magnetic fields alongside matter, but this is analogous to the $\vec{D}$ field that is a combination of some "driving," more fundamental force ($\vec{E}$ field) and its response from matter (the $\vec{P}$ field).

5.2.2 The magnetization field

Similar to the polarization field, we have the magnetization field, $\vec{M}$. Like polarization, this is the net of the dipole moments of the individual particles; unlike polarization, there are two types of magnetic dipole moments caused by atoms.

First of all, we need to define the magnetic moment $\vec{m}$:

$$\vec{m} = I\vec{A} \tag{88}$$

where I is some quantity of current, and $\vec{A}$ is a vector with magnitude the area or the surface enclosed by the loop and direction normal to the surface enclosed by the loop (similar to angular momentum). This can be applied both to electron orbits (if we consider the electron to be traveling in the classical sense of "orbiting" the nucleus), or in the quantum-mechanical sense of electron "spin."

The magnetization field is the bulk of the average magnetic fields:

$$\vec{M} = N\langle\vec{n}\rangle \tag{89}$$

Like the polarization field, it should be proportional to the magnetic field strength (assuming linear material):

$$\vec{M} = \chi_M\vec{H} \tag{90}$$

where χ_M is the magnetic susceptibility (the same conclusions can be drawn as for the electric susceptibility). Note the slight asymmetry from the polarization field: there is no multiplication by μ_0 here.

5.2.3 Revisiting the magnetic field

We provide a more precise formulation of the $\vec{B}$ field analogous to that of the displacement field:

$$\vec{B} = \mu_0\vec{H} + \mu_0\vec{M} = \mu_0(1 + \chi_M)\vec{H} \tag{91}$$

If we define the magnetic permeability of dielectric material:

$$\mu = \mu_0(1 + \chi_M) \tag{92}$$

(Note that for many materials, $\mu \approx \mu_0$, so this is a valid approximation.) Then the equation for $\vec{B}$ simplifies to:

$$\vec{B} = \mu\vec{H} \tag{93}$$

fitting a simple linear relationship analogous to the displacement field (assuming linear material).

5.2.4 Categories of magnetic materials

We can generally classify materials into one of three categories:

Paramagnetic: $\mu > \mu_0$, $\chi_M > 0$. Thus the magnetization occurs in the same direction as the imposed $\vec{H}$ field. This is due to a dominating magnetic moment caused by electron spin (i.e., many half-full orbitals). E.g., O_2, Al.

Diamagnetic: $\mu < \mu_0$, $\chi_M < 0$. Thus the magnetization opposes the $\vec{H}$ field. This is due to small contribution to magnetic moment due to electron spin (i.e., mostly full orbitals), and thus the electron "orbits" around the nucleus dominate the magnetic moment. E.g., H_2O, N_2, Cu.

Ferromagnetic: $\mu \gg 1$ (not really a subset of paramagnetic because of its more extreme properties). This is due to more complex (nonlinear) material, i.e., $\mu = \mu(H)$, above some critical temperature (the "curie temperature" T_{curie}). These exhibit hysteresis (can be interpreted as "memory" or "delay" in their response to changes in the $\vec{H}$ field). E.g., Fe (hence the name), Ni, Fe_3O_4.

5.3 Maxwell equations in material

Rearranging (Eq. 1), which still holds in material, we get:

$$\nabla \cdot \epsilon_0\vec{E} = \rho \tag{94}$$

Noting that $\epsilon - \epsilon_0 = \epsilon_0\chi_E$, we get:

$$\nabla \cdot (\epsilon - (\epsilon - \epsilon_0))\vec{E} = \nabla \cdot (\epsilon\vec{E}) - \nabla \cdot (\epsilon_0\chi_E\vec{E}) = \nabla \cdot \vec{D} - \nabla \cdot \vec{P} = \rho \tag{95}$$

Noting that ρ consists of "free charge" ρ_f (free electrons and those in conducting material, which conduct current when an electric field is applied) and "bound charge" ρ_b (from dielectric dipoles), then:

$$\rho = \rho_f + \rho_b \tag{96}$$

A hand-wavy proof of the Maxwell equation for the divergence of $\vec{D}$ in matter says that if the bound charge density comes from the divergence of the polarization field (dielectrics), then the free charge density is from the divergence of the displacement field (credit to `https://em.geosci.xyz/content/maxwell1\_fundamentals/formative\_laws/gauss\_electric.html`). (This isn't really a proof, but this result is true.) Then we get the following two equations:

$$\nabla \cdot \vec{D} = \rho_f \tag{97}$$

$$-\nabla \cdot \vec{P} = \rho_b \tag{98}$$

The former becomes our first Maxwell equation in material; the latter is another way to arrive at (Eq. 87).

The rest of the Maxwell equations are mostly the same, but can be rewritten using the constituent equations. We can assume that there are no free charges in dielectric material, so the Maxwell equations in dielectric material:

$$\nabla \cdot \vec{D} = \rho_f = 0$$

$$\nabla \cdot \vec{H} = \frac{1}{\mu}\nabla \cdot \vec{B} = \frac{1}{\mu}(0) = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The induction law is a little more complicated; see `http://www.oceanopticsbook.info/view/radiative_transfer_theory/level_2/maxwells_equations_in_matter` for a more in-depth explanation (it involves looking at the free, bound, and polarization currents); for our purposes, we can just think of it as an extension of the vacuum equation, replacing permittivity and permeability of free space with their material counterparts (this again assumes no free current):

$$\nabla \times \vec{B} = \mu\nabla \times \vec{H} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} = \mu\frac{\partial}{\partial t}(\epsilon\vec{E}) \Rightarrow \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

Summary of equations and constituent relations (in linear dielectric material with no free charge):

$$\nabla \cdot \vec{D} = 0 \tag{99}$$

$$\nabla \cdot \vec{B} = 0 \tag{100}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{101}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \tag{102}$$

$$\vec{D} = \epsilon \vec{E} \tag{103}$$

$$\vec{B} = \mu \vec{H} \tag{104}$$

This set of equations is more symmetric between the electric and magnetic fields, with the introduction of the "underlying" $\vec{H}$ field and "overlying" $\vec{D}$ field. To complete the picture, these cause some "response" in dielectric matter (the size of the response depends on the material's susceptibility), which are the $\vec{M}$ and $\vec{P}$ fields; combining the fundamental and material response fields leads us to the $\vec{B}$ and $\vec{D}$ fields.

As expected, these simplify to the vacuum Maxwell equations when in a vacuum, since the "response" fields are zero since the susceptibility of free space is zero.

A straightforward application of these new relations is to show another representation of the Poynting vector, which also nicely more symmetric once we only involve the fundamental fields:

$$\vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B} = \vec{E} \times \vec{H} \tag{105}$$

We will revisit the Maxwell relations in material, with the possibility of free charges and current, in the last section (Sec. 7.2).

5.4 Interface conditions

For these, we show the more general forms and the less general ones.

5.4.1 Pillbox approach for flux

We approach the first two Maxwell equations. First, don't assume no free charge. Imagine a small "pillbox" (or other prism) straddling the interface between two materials, the first with permittivity ϵ_1 and the second with permittivity ϵ_2, with its two flat ends (locally approximately) parallel to the interface, one flat end in each material, and the sides of pillbox of infinitesimal length. The total $\vec{E}$ flux through this surface is almost all from the two parallel interfaces, since the area of the sides is negligible, and all of it is normal to the interface.

For the displacement field, we have the general form (Eq. 97). Assuming $\hat{n}$ is the unit normal vector pointing from material 1 to material 2, the net flux through this is

$$\nabla \cdot \vec{D} = \hat{n} \cdot \vec{D}_2 + (-\hat{n}) \cdot \vec{D}_1 = \hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_f \tag{106}$$

Since this pillbox is essentially on the boundary of the material, this free charge is not actually the free charge volume density, but rather the free charge surface density, σ_f, so this may rewrite this as:

$$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma_f \tag{107}$$

Of course, assuming no free charge (as we do for this course), $\sigma_f = 0$, so:

$$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = 0 \tag{108}$$

This means that the normal component of the $\vec{D}$ field is continuous across a boundary (when there is no free charge), or that:

$$\vec{D}_1^{\perp} = \vec{D}_2^{\perp} \tag{109}$$

$$\epsilon_1 \vec{E}_1^{\perp} = \epsilon_2 \vec{E}_1^{\perp} \tag{110}$$

The same pillbox method can be used to obtain analogous relations for $\vec{B}$ across an interface (but this holds whether there is free charge or not):

$$\vec{B}_1^{\perp} = \vec{B}_2^{\perp} \tag{111}$$

$$\mu_1 \vec{H}_1^{\perp} = \mu_2 \vec{H}_2^{\perp} \tag{112}$$

5.4.2 Loop approach for curl

We can create a similar setup to the pillbox method, but instead of a pillbox, we have a rectangular loop: two sides (locally approximately) parallel to the surface, and the two other sides perpendicular to the surface and infinitesimally long. In a similar way to the pillbox method, we find the curl around the loop, neglecting that caused by the normal components, and apply Stoke's Theorem on (Eq. 101):

$$\oiint_A (\nabla \times \vec{E}) \cdot d\vec{s} = \oint_C \vec{E} \cdot d\vec{l} = -\oiint_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \tag{113}$$

This essentially transforms it back into the integral form. Since as we take the limit of the sides normal to the interface as their length goes to 0, the limit of the area goes to zero, so the enclosed $\vec{B}$ flux also goes to zero (the right side of the equation). The curl of $\vec{E}$ (the left side of the equation) is the sum of the $\vec{E}$ tangential to the interface in opposite directions; this magnitude of the tangential field can be expressed as $\hat{n} \times \vec{E}$. Thus:

$$\hat{n} \times \vec{E}_2 + (-\hat{n}) \times \vec{E}_1 = \hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \tag{114}$$

This means that the tangential part of the $\vec{E}$ field is continuous. The same result can be shown for $\vec{H}$ fields using the fourth Maxwell equation; this gives us the following tangential interface conditions:

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \tag{115}$$

$$\frac{1}{\epsilon_1}\vec{D}_1^{\parallel} = \frac{1}{\epsilon_2}\vec{D}_2^{\parallel} \tag{116}$$

$$\vec{H}_1^{\parallel} = \vec{H}_2^{\parallel} \tag{117}$$

$$\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel} \tag{118}$$

5.4.3 Example: refraction of a (steady) electric field

These boundary conditions describe how static fields act near boundaries; this describes refraction. If α describes angle from the normal of an $\vec{E}$ field vector as near a horizontal interface, then from the normal and tangential matching conditions, respectively, we have:

$$\epsilon_1 E_1 \cos\alpha_1 = \epsilon_2 E_2 \cos\alpha_2 \tag{119}$$

$$E_1 \sin\alpha_1 = E_2 \sin\alpha_2 \tag{120}$$

Combining these two:

$$\frac{\tan\alpha_1}{\epsilon_1} = \frac{\tan\alpha_2}{\epsilon_2} \tag{121}$$

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6 Reflection and transmission

This section is an application of the boundary conditions derived in the previous section, but now we apply it to plane waves (as opposed to static fields). We deal first with the simpler case of normal incidence (when the wave vector is normal to the interface) and then extend the results to oblique incidence.

6.1 Index of refraction, propagation speed, and impedance

From the Maxwell equations in material, we get:

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \tag{122}$$

Which is an extension of the vacuum case (clearly, this agrees with the vacuum case since $v = c$ if $\mu = \mu_0$ and $\epsilon = \epsilon_0$). The index of refraction is defined as the ratio of c to v:

$$n = \frac{c}{v} = \frac{\frac{1}{\sqrt{\mu_0\epsilon_0}}}{\frac{1}{\sqrt{\mu\epsilon}}} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\kappa} \tag{123}$$

Usually, $n > 1$, but it is possible to have n be less than unity: see `https://en.wikipedia.org/wiki/Refractive_index#Refractive_index_below_unity`. Now, define the impedance Z of a material to be:

$$Z = \sqrt{\frac{\mu}{\epsilon}} \tag{124}$$

Note that this has units of electrical resistance ("impedance"). Plugging this into (Eq. 38), we can rewrite the relationship between $\vec{B}$ and $\vec{E}$ as:

$$E = \frac{\omega}{k}(\mu H) = \frac{\mu}{\sqrt{\mu\epsilon}} H = ZH \tag{125}$$

and thus we can rewrite Z as:

$$Z = \frac{E}{H} \tag{126}$$

Also, note that Z is inversely proportional to n (and thus directly proportional to v). (It may be a little counterintuitive that a higher impedance corresponds to a higher wave speed; this can be thought of like in a string wave, in which a wave moves faster the tauter the string is (which can be thought of as having a higher "impedance")).

$$Z = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu}{\epsilon}\frac{\mu}{\mu}} = \sqrt{\frac{\mu^2}{\mu\epsilon}} = \mu v = \mu\frac{c}{n} \Rightarrow Z \propto n^{-1} \tag{127}$$

6.2 Fresnel equations

(For this section, refer to the notes for diagrams.)

There are two polarization cases: the first, s-polarization (a.k.a., TE, transverse electric), when $\vec{E}$ is parallel to the interface (and normal to the incidence plane); and p-polarization (a.k.a., TM, transverse magnetic), when $\vec{E}$ is in the incidence plane. This is the general oblique case, for which normal incidence can be derived when $\theta_i = 0$.

Note that while s- and p-polarization are specific polarization cases, they are orthogonal and span the polarization plane (which is two-dimensional); thus, any EM wave with any polarization can be represented as a linear combination of the two. Also, there are a few assumptions made about the directions of the reflected and transmitted waves that are not explained; these are due to solving all of the interface conditions simultaneously.

Setup: the interface is the (horizontal) plane $z = 0$, the incidence plane is the plane $x = 0$. The incident wave has a wave vector aimed toward the origin from the second quadrant of the y-z plane, i.e.,

$$\vec{k}_i = k_i \sin\theta_i \hat{y} - k_i \cos\theta_i \hat{z} \tag{128}$$

$$\vec{k}_r = k_r \sin\theta_r \hat{y} + k_r \cos\theta_r \hat{z} \tag{129}$$

$$\vec{k}_t = k_t \sin\theta_t \hat{y} - k_t \cos\theta_t \hat{z} \tag{130}$$

Note that all three vectors lie in the incidence plane; see (Sec. 6.2.2).

6.2.1 s-polarization case

In this case, $\vec{E}$ vectors are normal to the plane of incidence ("s" for "perpendicular" (in German)), and $\vec{B}$ vectors lie in the plane of incidence, with $\vec{B}_i$ pointing NE, $\vec{B}_r$ pointing SE, and $\vec{B}_t$ pointing NE. Assume $\vec{E}_i$ and $\vec{E}_t$ point into the sheet of paper, and $\vec{E}_r$ points out of the sheet of paper.

(N.B. These signage decisions are arbitrary but important to keep consistent through a proof; note that we keep right-handed triads of $\vec{E}$ and $\vec{B}$ consistent, and thus their magnitudes can go negative as long as the signs of the $\vec{E}$ and $\vec{B}$ vectors match: i.e., $E_x/|E_x| = B_x/|B_x|$. [In this orientation, as $\theta_i = 0$, the signs of $\vec{E}_i$ and $\vec{E}_r$ are opposite, which is consistent with the setup for the normal case; we could have changed the orientation so that $\vec{E}_i$ and $\vec{E}_r$ would be in the same direction if $\theta_i = 0$, but the magnitude would be inverted. Either way, it makes no difference to the reflection or transmission (power) coefficients, which are the square of the magnitudes of the reflected and transmitted waves, respectively.] However, note that we are asserting that these are the orientations of the reflected and transmitted $\vec{E}$ and $\vec{B}$ vectors (i.e., that the $\vec{E}$ waves stay parallel to the plane of incidence and that the $\vec{B}$ waves stay in the plane of incidence) without proof, and using these to solve for the magnitudes assuming they're in this orientation. I have no idea how to prove that these are the correct orientations (and can't seem to find it online), but assume it is some problem of simultaneously solving all of the boundary conditions at once (including the normal ones, which we don't deal with here), and/or some calculus/optimization problem.)

Then, on the boundary:

$$\vec{E}_i = \vec{E}_{i_0} e^{i(yk_i \sin\theta_i - \omega_i t)} \tag{131}$$

$$\vec{E}_r = \vec{E}_{r_0} e^{i(yk_r \sin\theta_r - \omega_r t)} \tag{132}$$

$$\vec{E}_t = \vec{E}_{t_0} e^{i(yk_t \sin\theta_t - \omega_t t)} \tag{133}$$

Note here that $\vec{k} \cdot \vec{r}$ simplifies to only having a $\hat{y}$ component, since $k_x = 0$ and $z = 0$ (on the boundary). Then, apply the tangential boundary conditions:

$$\vec{E}_i^{\parallel} + \vec{E}_r^{\parallel} = \vec{E}_t^{\parallel} \tag{134}$$

$$\vec{H}_i^{\parallel} + \vec{H}_r^{\parallel} = \vec{H}_t^{\parallel} \tag{135}$$

In the case of s-polarization, the $\vec{E}$ vectors are already parallel to the plane, but the $\vec{B}$ are not. Thus, for s-polarization, we can write these more concretely as:

$$-E_{i_0} e^{i(yk_i \sin\theta_i - \omega_i t)} - E_{r_0} e^{i(yk_r \sin\theta_r - \omega_r t)} = -E_{t_0} e^{i(yk_t \sin\theta_t - \omega_t t)} \tag{136}$$

$$H_{i_0} \cos\theta_i e^{i(yk_i \sin\theta_i - \omega_i t)} + H_{r_0} \cos\theta_i e^{i(yk_r \sin\theta_r - \omega_r t)} = H_{t_0} \cos\theta_i e^{i(yk_t \sin\theta_t - \omega_t t)} \tag{137}$$

6.2.2 The laws of reflection and refraction

We make the observation that this set of equations must be true for all moments in time and all points in space. Thus, at any given point in space, in order to satisfy both boundary equations, all of the time terms must factor out. In other words, two sinusoids cannot sum to another sinusoid unless their frequencies are equal, and the sum sinusoid must have the same frequency. Thus they must all be oscillating at the same rate in time:

$$e^{i(-\omega_i t)} = e^{i(-\omega_r t)} = e^{i(-\omega_t t)} \Rightarrow \omega_i = \omega_r = \omega_t \tag{138}$$

Note that since this is a simple result of sinusoid math and does not involve the boundary conditions, it is true not only for EM waves, but other waves at boundaries (e.g., waves on a string). By the same logic, we can also deduce that the oscillations in space must also be due to an equal space-phase:

$$e^{iyk_i \sin\theta_i} = e^{iyk_r \sin\theta_r} = e^{iyk_t \sin\theta_t} \Rightarrow k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t \tag{139}$$

Since $k = \omega/v$, and we know from (Eq. 138) that $\omega_i = \omega_r$, and $v_i = v_r = v_1$ since waves travel at one speed through the one medium dependent on its index of refraction, then $k_i = k_r = k_1$. (Thus, it makes sense to define $k_2 := k_t$, and $\theta_2 := \theta_t$.) This allows us to draw two conclusions (both of which are general to waves at an interface, again without the EM boundary conditions). Firstly, the law of reflection:

$$\theta_i = \theta_r (:= \theta_1) \tag{140}$$

and then the law of refraction (Snell's law)

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \tag{141}$$

6.2.3 Aside: $\vec{k}$ lying in the incidence plane

With these relations and this logic, we can now show that all of the $\vec{k}$ vectors lie in the incidence plane. We assumed that $k_{r_x} = k_{t_x} = 0$; for now, assume they are not. Then, assuming that $\vec{k}_r$ and $\vec{k}_t$ lie ψ_r and ψ_t off of the normal axis (and off the incidence plane) in the $\hat{x}$-direction, the space-phase component ϕ_s of the three vectors are:

$$\begin{aligned} \phi_{i_s} &= iyk_i \sin\theta_i \\ \phi_{r_s} &= ik_r(y\sin\theta_r + x\sin\psi_r) \\ \phi_{t_s} &= ik_t(y\sin\theta_t + x\sin\psi_t) \end{aligned}$$

By the same logic as above, these three space-phases must be equal at all points in the $\hat{x}$ and $\hat{y}$ directions at any point in time (the same logic applies for the periodicity of the $\hat{x}$ and $\hat{y}$ dimensions in a complex sinusoid as it does for the t dimension). Thus these must work when $x \neq 0$ and $y = 0$, in which case setting all of the equations equal yields:

$$0 = x\sin\psi_r = x\sin\psi_t \tag{142}$$

Thus forcing $\sin\psi = 0$, so $\vec{k}$ must lie on the incidence plane.

6.2.4 Results of the s-polarization case

With the law of reflection and refraction, our boundary conditions get simplified to:

$$-E_{i_0} + E_{r_0} = -E_{t_0} \tag{143}$$

$$H_{i_0}\cos\theta_1 + H_{r_0}\cos\theta_2 = H_{t_0}\cos\theta_2 \tag{144}$$

Some magic/algebra happens here... We obtain the s-polarization reflectivity and transmission (amplitude) coefficients:

$$r_s = \frac{E_{r_0}}{E_{i_0}} = -\frac{\cos\theta_i - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}} \tag{145}$$

$$t_s = \frac{E_{t_0}}{E_{i_0}} = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}} \tag{146}$$

6.2.5 p-polarization case setup and results

In this case, the $\vec{E}$ fields lie in the plane of incident ("p" for "parallel" (also in German)). Imagine that the $\vec{k}$ vectors are situated the same way, all three $\vec{B}$ vectors point straight out of the page toward you, and $\vec{E}_i$ points in the NE direction, $\vec{E}_r$ points in the NW direction, and $\vec{E}_t$ points in the NE direction. Keep in mind the N.B. from the s-polarization section. Again, we use the tangential boundary conditions; here, $\vec{B}$ is already tangential to the interface, but $\vec{E}$ is not. The results are very similar (including the law of reflection, law of refraction, and the wave vectors in the plane of incidence), so we will skip much of the algebra and skip to the results. We have the p-polarization reflectivity and transmission (amplitude) coefficients:

$$r_p = \frac{E_{r_0}}{E_{i_0}} = -\frac{\left(\frac{n_2}{n_1}\right)^2\cos\theta_i - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_i + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}} \tag{147}$$

$$t_p = \frac{E_{t_0}}{E_{i_0}} = \frac{2\frac{n_2}{n_1}\cos\theta_i}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_i + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}} \tag{148}$$

6.3 Normal incidence Fresnel equations

We can derive these simply from the more complex oblique cases derived above. The orientations match the limit of the orientations for both the s- and p-polarization case; i.e., the $\vec{E}_r$ is antiparallel to $\vec{E}_i$ and $\vec{E}_t$; and all of the $\vec{B}$s are parallel and pointing in the same direction. With $\theta_i = 0$, the two polarization cases converge to the same result. We have the normal case reflectivity and transmission (amplitude) coefficients:

$$r_\perp = \frac{n_2 - n_1}{n_2 + n_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{149}$$

$$t_\perp = \frac{2n_1}{n_2 + n_1} = \frac{2Z_2}{Z_1 + Z_2} \tag{150}$$

The relation to Z is due to (Eq. 127). What this means is that if the index of refraction of material 2 is greater than that of material 1 (or, alternatively, if the impedance of material 1 is greater than that of material 2), then the reflected $\vec{E}$ will be antiparallel; and vice versa. If the materials have the same index of refraction, then there will be zero reflection and total transmission (this is called impedance matching). The transmission, however, will always be parallel to the incident wave.

6.4 Reflection and transmission power coefficients

We can define the reflection and transmission power coefficients:

$$R \coloneqq \frac{\langle \vec{S}_r \rangle \cdot \hat{n}}{\langle \vec{S}_i \rangle \cdot \hat{n}} \tag{151}$$

$$T \coloneqq \frac{\langle \vec{S}_t \rangle \cdot \hat{n}}{\langle \vec{S}_i \rangle \cdot \hat{n}} \tag{152}$$

In other words, R and T are the power flux coefficients. Extending (Eq. 44) to use ϵ and μ, we have:

$$\langle \vec{S} \rangle = \frac{1}{2Z} E_0^2 \hat{k} \tag{153}$$

and thus R and T can be simplified to (using Snell's law and the fact that $Z_i = Z_r \neq Z_t$):

$$R = \left| \frac{E_{r_0}}{E_{i_0}} \right|^2 \tag{154}$$

$$T = \left| \frac{E_{t_0}}{E_{i_0}} \right|^2 \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} \tag{155}$$

6.5 Evanescent waves

We define the critical angle to be:

$$\theta_{crit} \coloneqq \sin^{-1}\left(\frac{n_2}{n_1}\right) \tag{156}$$

When $\theta_i < \theta_{crit}$, then the transmitted amplitude coefficient t is real. When $\theta_i = \theta_{crit}$, then there is no transmission $t = 0$ and total internal reflection

($r = 1$). However, when $\theta_i > \theta_{crit}$, then $r \in \mathbb{C}$ (but $|r| = 1$ and $t = 0$, so it is still total internal reflection). This in and of itself is not very revealing, so just let the math speak for itself.

$$\vec{E}_t = \vec{E}_{t_0} e^{i(\omega_t - k_2(y \sin\theta_2 + z\cos\theta_2))} \tag{157}$$

Making the substitutions (cosine identity, Snell's law, definition of critical angle) and the fact that $\sin\theta_1 > \sin\theta_{crit}$:

$$\cos\theta_1 = \pm\sqrt{1 - \sin^2\theta_1} \tag{158}$$

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 \tag{159}$$

$$\frac{n_2}{n_1} = \sin\theta_{crit} \tag{160}$$

then we can rewrite the transmitted $\vec{E}$ wave as:

$$\vec{E}_t = \vec{E}_{t_0} e^{-k_2 z\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{crit}}\right)^2 - 1}} e^{i\omega t - ik_2 y \frac{n_2}{n_1}\sin\theta_1} \tag{161}$$

Note here that the wave number k (what is being dotted with the position vector) becomes complex; the complex part gets multiplied with i again in the exponent to become real. (This will come up again in the next two topics.) We can define the folding distance z_0 to be:

$$z_0 = \frac{1}{k_2\sqrt{\left(\frac{\sin\theta_1}{\sin\theta_{crit}}\right)^2 - 1}} \tag{162}$$

(in general, the folding distance is z_0 s.t. the solution has an $\exp(z/z_0)$ coefficient) and trivially solve for the velocity and wave number of this wave:

$$v = \frac{\omega}{k_2\left(\frac{n_1}{n_2}\right)\sin\theta_1} \tag{163}$$

$$k = k_2\left(\frac{n_1}{n_2}\right)\sin\theta_1 \tag{164}$$

to rewrite this wave as:

$$\vec{E}_t = \vec{E}_{t_0} e^{-\frac{z}{z_0}} e^{ik(y - vt)} \tag{165}$$

This is an electric wave traveling parallel to the interface with exponentially-decreasing magnitude the further you get away from the interface. This is called the "evanescent wave," and it is traveling in the denser material 2, which is now called the "forbidden region." If you get very close to the surface, then you can detect or interact with this wave; this is called "frustrated total internal reflection" (FTIR); otherwise, this energy just travels next to the surface and is not lost. In the case of FTIR, then the reflection is not total; even though the wave is hitting the interface at an angle greater than the critical angle, this is a way to still transmit (lose) energy across the interface.

It is also sometimes referred to as "evanescent wave coupling," as messing with the evanescent wave will change the totality of the reflection, and is analogous to a quantum-mechanical phenomenon (quantum tunneling of wave functions; according to Wikipedia).

6.6 Brewster's angle

Now that we've found the case for zero transmission ($\theta = \theta_{crit}$) and zero-ish transmission ($\theta > \theta_{crit}$), what about the case for zero reflection? $r = 0$ only occurs in the p-polarization case:

$$r_p = 0 \Rightarrow \left(\frac{n_2}{n_1}\right)^2 \cos\theta_1 = \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_1} \tag{166}$$

Define the angle that satisfies this relation to be Brewster's angle, θ_B:

$$\theta_B \coloneqq \tan^{-1}\left(\frac{n_2}{n_1}\right) \tag{167}$$

At this angle, the angle between $\hat{k}_r \perp \hat{k}_t$. This geometry is relevant; the zero transmission is due to the anisotropic nature of the Larmor power radiation; at this angle of incidence, electrons will accelerate in a motion parallel to $\hat{k}_r$ and normal to $\hat{k}_t$, thus not radiating any in the direction of the reflected wave. This won't happen in the s-polarization case since the $\vec{E}$ wave is parallel to the interface, and thus will be accelerating electrons back and forth in the interface plane, so the region of no radiation will be in the interface plane and thus cannot be in the direction of the reflected wave.

In the case of random polarization (a mix of s- and p-polarization) hitting a surface at Brewster's angle, this means that the reflected light will be s-polarized and the transmitted light will be p-polarized; this can be used as a simple polarization filter and has its applications with camera glare (if the light reflected from a surface is heavily s-polarized, then blocking out that polarization of light will greatly reduce the reflection (glare) from that surface while affecting less of the randomly-polarized light from other materials).

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7 Absorption and dispersion in dielectrics

In dielectrics, we will derive a frequency-dependent index of refraction, n. This means that light waves at different frequencies will move at different speeds through a material, and will reflect/refract differently; e.g., a rainbow or white light going through a prism are examples of dispersion. This is closely related to another phenomenon, absorption; we will see that with complex n values, there is some absorption of light in matter; because of dispersion, different frequencies cause different n values, which in turn cause differing amounts of absorption, which lead to our concept of the color of materials based on the intrinsic properties of the material (the fundamental frequencies of its molecules).

7.1 Improved spring model

Recall that the moment of a dipole is:

$$\vec{p} = q\vec{d} \tag{168}$$

and recall that in a dielectric material, $\vec{d}$ (charge separation) is represented by the motion of an electron on a spring, which is dependent on the fundamental frequency of the charge. In the previous spring model, we assumed that a particle only has one fundamental frequency. Let us consider all of the fundamental frequencies of an atom, represented by some distribution; let i denote the number of different characteristic frequencies of an atom, $\vec{p_i}$ denote the dipole moment of the i-th frequency of an atom, and f_i be the fraction of electrons within that atom with that fundamental frequency (i.e., $f_i = f(\omega_{i_0})$ is the p.m.f. of the fundamental frequencies). Then the polarization field may be generalized to the bulk polarization of this sum of dipole moments (contrast this with (Eq. 81)):

$$\vec{P} = \sum_i N f_i \vec{p}_i \tag{169}$$

We have already solved for the solution for a single electron, and this more general case is simply the sum (recall that $\vec{p} = q\vec{x}$, since the displacement of an electron is equal to its dipole moment):

$$\vec{P} = \sum_i \frac{N f_i q^2}{m((\omega_{0_i} - \omega)^2 + j\gamma_i\omega)} \vec{E}_{inc} \tag{170}$$

(Here we switch the convention to representing the pure imaginary number as $j = \sqrt{-1}$ to avoid confusion with the indexing variable.) For linear dielectric materials, we can rearrange (Eq. 82) to get:

$$\chi_e = \frac{\vec{P}}{\epsilon_0 \vec{E}} \tag{171}$$

We also know from (Eq. 123) that $n = \sqrt{\kappa}$ and from (Eq. 84) that $\kappa \coloneqq 1 + \chi_e$, so we can solve for n:

$$n^2(\omega) = 1 + \frac{1}{\epsilon_0} \sum_i \frac{N f_i q^2}{m((\omega_{0_i} - \omega)^2 + j\gamma_i\omega)} \tag{172}$$

Thus, n is the square root of a complex number and is thus itself complex (we ignore the technicalities of there being two square roots, just focus on one). Since $n = \frac{c}{\omega}k$, and c and ω are real constants (don't know how to interpret complex speed or frequency), this means that k is also complex. In particular, we see that for an EM wave passing through dielectric material, the amplitude of the wave decreases exponentially with distance if k has an imaginary component:

$$\vec{E} = \vec{E}_0 e^{-\Im(\vec{k}\cdot\vec{r})} e^{i(\Re(\vec{k}\cdot\vec{x})-\omega t)} \tag{173}$$

In other words, the real part of the wave number is like the ordinary wave number (the "space-frequency" of the wave), while the imaginary part represents damping (this is seen in the evanescent wave case; the wave number in that case is complex). Thus, if we wanted to solve for the damping coefficient, we could solve for $\Im(k)$:

$$\Im[k(\omega)] = \frac{\omega}{c}\Im[n(\omega)] \tag{174}$$

We can see from the equation for $n(\omega)$ that if $(\omega_{0_i} - \omega)^2 \gg 0$, then the numerator and denominator are mostly real (and positive), and the resulting $n^2(\omega)$ is mostly real. However, as $\omega_{0_i} - \omega \to 0$, then we have a real numerator divided by a mostly imaginary denominator, resulting in an imaginary quotient (and thus larger damping). Thus, we have damping (absorption) that is largest at the fundamental frequencies of the atom.

The frequencies that correspond to absorption peaks (peaks of $\Im(n)$ and $\Im(k)$) then correspond to our idea of color: the stronger the absorption, the weaker the reflection of any particular frequency of light.

7.2 A better model: $\vec{E}_{site}$ and the Clausius-Mossotti relation

Note that the previous equation for n works best for gases and other sparse dielectrics, since in denser materials we have external contributions from the $\vec{p}$ of other molecules; this model (i.e., the solution to the damped oscillator) assumes that the primary driving force is some external oscillator $\vec{E}_{ext}$ wave. A better model takes into account the imposed electric field. We define the electric field in a hole (vacuum) surrounded by uniform dielectric material as $\vec{E}_{site}$. Dependent on the shape and orientation of the hole, the boundary conditions cause the imposed $\vec{E}_{ext}$ field to be different. For example, in a long thin slot parallel to the $\vec{E}_{ext}$ field, then:

$$\vec{E}_{site} \approx \vec{E}_{ext} \tag{175}$$

Since the dominating boundary is parallel to the $\vec{E}_{ext}$ and $\vec{P}$ fields, there is almost no polarization field lines. If the slit is in the plane normal to the $\vec{E}_{ext}$ field, then the polarization field is strong everywhere in the narrow slit:

$$\vec{E}_{site} \approx \vec{E}_{ext} + \frac{1}{\epsilon_0}\vec{P} \tag{176}$$

In a spherical hole (a result we won't derive):

$$\vec{E}_{site} \approx \vec{E}_{ext} + \frac{1}{3\epsilon_0}\vec{P} \tag{177}$$

This is a better approximation for the actual external (driving) $\vec{E}$ field on an atom in a dielectric material. Using this result, we obtain the Clausius-Mossotti relation to more accurately calculate n (implicitly) in denser dielectric material:

$$\frac{n^2-1}{n^2+2} = \frac{Nq^2}{3\epsilon_0 m}\sum_i \frac{f_i}{(\omega_{0_i}-\omega)^2 + j\gamma_i\omega} \tag{178}$$

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8 EM waves in conductors

8.1 Free charges and currents in Maxwell equations

We have dealt with dielectric material for a long time, and been able to make assumptions such as that $\rho_f = \vec{j}_f = 0$, which simplified the Maxwell equations. However, in conductors, this is not true anymore. In general, we can express current in this version of Ohm's law (i.e., integrating this over space gives us Ohm's law):

$$\vec{j}_f = \sigma\vec{E} \tag{179}$$

where σ is the conductance of a material (inversely proportional to resistance). The full Maxwell equations are in effect here (these were all stated previously in (Sec. 5.2.4), except we the current component of Ampere's law was disregarded because of no free charges in dielectrics):

$$\nabla\cdot\vec{D} = \rho_f \tag{180}$$

$$\nabla\cdot\vec{H} = 0 \tag{181}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{182}$$

$$\nabla\times\vec{H} = \sigma\vec{E} + \frac{\partial\vec{D}}{\partial t} \tag{183}$$

We can start with the continuity equation (local conservation law) for charges (this is similar to any other conservation law, such as that for power flux/energy (Eq. 43)):

$$\frac{\partial\rho_f}{\partial t} + \nabla\cdot\vec{j}_f = 0 \tag{184}$$

and, by substituting Ohm's law and the general form of Gauss's electric law, we obtain:

$$\rho_f(t) = \rho_{f_0}\exp\left(-\frac{\sigma}{\epsilon}t\right) \tag{185}$$

where $\rho_{f_0} = \rho_f(0)$. Thus any free charge density in a conducting material will dissipate at an exponential rate, with the time constant $\tau = \epsilon/\sigma$ being inversely proportional to the material's conductance and directly proportional to its permittivity. For good conductors with high conductivity, $\tau \approx 10^{-19}$, so charges dissipate very quickly. Because of this, we can approximate that

$\rho_f \approx 0$ at any moment of time in a conductor, so we can leave the first Maxwell equation for conductors as we have seen it:

$$\nabla \cdot \vec{D} \approx 0 \tag{186}$$

8.2 Dispersion relation

We can solve for the wave equation like we did earlier in the semester:

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}\right)$$

$$\nabla^2 \vec{E} = \mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2} \tag{187}$$

The same form can be derived for $\vec{B}$ (or $\vec{H}$). If we substitute the general form for a plane wave solution:

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{r} + \phi_0)}$$

into this differential relation for $\vec{E}$, then by our rules of differential operators on plane waves (Sec. 1.7):

$$-k^2\vec{E} = \mu\sigma(i\omega)\vec{E} + \mu\epsilon(-\omega^2)\vec{E}$$

$$k^2 = \mu\epsilon\omega^2 - i\mu\sigma\omega \tag{188}$$

This is a disperson equation: from Wikipedia, "a dispersion relation relates the wavelength or wavenumber of a wave to its frequency." Here we have again a complex wave number, just like in the case for the evanescent wave and imaginary wave numbers in dielectrics. (Note that as $\sigma \to 0$, then $k = \Re(k)$, and we approach the simple real k case.) Similarly, this means that we have damping/absorption in a metal, and as before folding distance is:

$$z_0 = \frac{1}{\Im(k)} \tag{189}$$

(This particular folding distance is called the "skin depth" of a conductor.) Since $\Im(k)$ is small, then the skin depth is usually pretty short, which means that the EM fields inside of a conductor are near zero (which agrees with our assumptions/observations last semester in PH213).

8.3 Additional notes: phase shifts and reflectivity

Also note that, for complex wave numbers, there is an additional phase shift between $\vec{E}$ and $\vec{B}$, which does not occur in the real case. In particular, if we write $\vec{k}$ as a complex vector $\left(|k|e^{i \arg k}\right)\hat{k}$ with Faraday's law applying our differential operators:

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ -i\vec{k} \times \vec{E} = -i\left(|k|e^{i \arg k}\right)\hat{k} \times \vec{E} &= -i\omega\vec{B} \\ |k|\hat{k} \times \vec{E} &= \omega\vec{B}e^{-i \arg k} \end{aligned} \tag{190}$$

This means that $\hat{k} \times \vec{E}$ is equal to some phase-shifted $\vec{B}$. (Again, this is also the case for evanescent and dielectric damping, but we choose to introduce it here, as it is not as important as previous results.)

A final comment is that with free charges, we may also have surface free charges and surface free currents. In this case, a good conductor has a very high reflection coefficient, which makes it a good reflector (mirror). (Not derived here but involves rewriting boundary conditions involving Ampere's law with free currents.)