MA345 – Test 2 Review

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This test covers Chapter 2: Elementary Functions, and Chapter 3: Integrals of *Complex Variables and Applications, 9th edition* by Churchill and Brown.

Contents

1 The exponential and log functions

1.1 Definitions

$$e^z := e^x e^{iy}$$

Here, if x is a root $1/n$, then the *positive* nth root is used, i.e., $e^{x=1/n} = \sqrt[n]{e}$. If $w = \rho e^{i\phi} = e^z$, then $\log w = z$. We define the inverse to be the log function:

$$\log w = \log(\rho e^{i\phi}) := \ln |z| + i \arg z = \ln \rho + i(\phi + 2n\pi),\ n \in \mathbb{Z}$$

1.2 Notes

- For calculating the exponential, we begin in rectangular form and end in polar form; the reverse is true for the logarithm.
- e^z is many-to-one and $\log z$ is one-to-many. Thus $e^{\log z} = z$ is a single value and $\log e^z = z + 2n\pi,\ n \in \mathbb{Z}$ is multi-valued. We may choose a single value of the log function by selecting a particular branch of the arg function, in particular, the principal logarithm $\operatorname{Log} z$ is defined as:

  $$\operatorname{Log} z := \ln |z| + i \operatorname{Arg} z$$

  and this reduces to the real case when z is a positive real.
- The complex exponential is defined over the entire complex plane and is entire; the logarithm is defined over the punctured complex plane. Since we need to choose a branch of the logarithm to get a singly-valued function, the logarithm is not analytic anywhere on its branch cut or at the origin (the branch point); however, it is analytic everywhere else.
- Because of branches and the "wrapping" nature of the exponential and logarithm, some properties of the exponential and log may not hold true everywhere, e.g., $\log z^c = c \log z$ is not true for all values of z, c and all branch cuts.

1.3 Properties

$$|e^z| = e^x$$

thus $|e^z| \neq 0$.

$$\arg e^z = y + 2n\pi,\ n \in \mathbb{Z}$$

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

$$\frac{1}{e^z} = e^{-z}$$

$$\frac{d}{dz} e^z = e^z$$

$$e^{z+2\pi in} = e^z,\ n \in \mathbb{Z}$$

thus e^z is $2\pi i$-periodic and is many-to-one.

$$\frac{d}{dz}\text{Log}\, z = \frac{1}{z}\ (z \neq 0, \text{Arg}\, z \neq \pi)$$

This uses the polar form of the C-R equations (since it is a pain to express the angle in terms of rectangular coordinates). The same is true for any branch, applying the appropriate condition for its branch cut.

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

$$\log \frac{z_1}{z_2} = \log z_1 - \log z_2$$

These properties are not always true for the principal branch.

2 The power function

2.1 Definitions

$$z^c := e^{c \log z}$$

where $c \in \mathbb{C}$ is some constant. In general, z is multiple-valued unless $c \in \{0, 1, 2, \dots\}$. The principal value of z^c is as expected:

$$\text{P.V.}\, z^c := c^{\text{Log}\, z}$$

We can use this to define the exponential function with base $c \in \mathbb{C}$:

$$c^z := e^{z \log c}$$

and has a principal value as expected.

2.2 Notes

- $z^n,\ n \in \mathbb{Z}$ and $z^{1/n},\ n \neq 0 \in \mathbb{Z}$ agrees with the previous definition (i.e., repeated multiplication and square roots). The former is singly-valued.
- Since this definition of the exponential function with arbitrary base suggests that exponential functions are multiply-valued, this suggests that the fundamental exponential is also multiply-valued. Thus, our usual interpretation of e^z is the principal exponential function with base e.
- Some real properties may not work here because of wrapping, e.g., $(z_1 z_2)^c = z_1^c z_2^c$ is not a valid identity for all z_1, z_2, and c.

2.3 Properties

$$\frac{1}{z^c} = z^{-c}$$

Again, we may have branches of the power function determined by the branches of $\log z$. To find the derivative, we must choose a particular branch of $\log z$, and then by the Chain rule:

$$\frac{d}{dz} z^c = cz^{c-1}$$

If we choose a specific value of $\arg c$, then c^z is entire, and its derivative is

$$\frac{d}{dz} c^z = c^z \log c$$

3 The trigonometric and hyperbolic functions

3.1 Definitions

The real sine and cosine functions can be defined using the exponential function and Euler's formula alone. We can define their complex analogues by replacing the real parameters with complex ones:

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}$$

The other trigonometric functions are defined in terms of these two functions in the same way. Their derivatives are all in the same form as their real counterparts.

We define the hyperbolic functions as their analogues are defined:

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

Likewise, we define the rest of the hyperbolic functions in the same way, and get results of the same form as their real counterparts.

We can obtain the inverses of the trigonometric and hyperbolic functions by solving a quadratic expression, where the unknown variable is e^{iz} or e^z. We get the following results:

$$\sin^{-1} z = -i \log \left[iz + (1 - z^2)^{1/2} \right]$$

$$\cos^{-1} z = -i \log \left[z + i(1 - z^2)^{1/2} \right]$$

$$\tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}$$

$$\sinh^{-1} z = \log \left[z + (z^2 + 1)^{1/2} \right]$$

$$\cosh^{-1} z = \log \left[z + (z^2 - 1)^{1/2} \right]$$

$$\tanh^{-1} z = \frac{1}{2} \log \frac{1 + z}{1 - z}$$

3.2 Notes

- As expected, the trigonometric and hyperbolic functions reduce to their real counterparts when $z \in \mathbb{R}$.
- While these functions are periodic and bounded in the real direction for a particular y, they are unbounded in the imaginary direction (and grow exponentially in magnitude).
- The inverse functions are multiply-valued, unless a single branch of the logarithm and square root are used.

3.3 Properties

$$\frac{d}{dz} \sin z = \cos z, \;\; \frac{d}{dz} \cos z = -\sin z$$

$$\sin -z = -\sin z, \;\; \cos -z = \cos z$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\sin \left(z + \frac{\pi}{2} \right) = \cos z, \;\; \sin \left(z - \frac{\pi}{2} \right) = -\cos z$$

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z+2\pi) = \sin z, \; \sin(z+\pi) = -\sin z, \; \cos(z+2\pi) = \cos z, \; \cos(z+\pi) = -\cos z$$

We can express the real and imaginary parts w.r.t. the trigonometric and hyperbolic functions of real variables:

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Note that to derive the above expressions, we use the sum identity on $z_1 = x$, $z_2 = iy$ and then differentiate the sin decomposition to achieve the cos decomposition. These identities can also be used to show:

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

$$\sin z = 0 \iff z = n\pi\, n \in \mathbb{Z}$$

$$\cos z = 0 \iff z = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$$

i.e., the zeros of the complex sine and cosine are the same as those of their real analogues. (And the zeros of the hyperbolic functions are of the same magnitude, but on the imaginary axis.)

$$\sinh iz = i \sin z,\ \cosh iz = \cos z$$

$$\sin iz = i \sinh z,\ \cos iz = \cosh z$$

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}$$

$$\frac{d}{dz}\cos^{-1} z = -\frac{1}{(1-z^2)^{1/2}}\ \left[= -\frac{d}{dz}\sin^{-1} z\right]$$

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$$

4 Derivatives and definite integrals of complex-valued functions of a real variable

4.1 Prelimary results

Let

$$w(t) = u(t) + iv(t)$$

where $t \in \mathbb{R}$, and u and v are real-valued. Then, when u' and v' exist, then the derivative $w'(t)$ is

$$w'(t) = u'(t) + iv'(t)$$

In other words, this shows us how to differentiate a function if we parameterize it w.r.t. a real variable. Since u and v are real-valued, we may integrate over them. Thus, the definite integral of w is something we already know from regular calculus:

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

This is well-defined whenever u and v are piecewise-continuous, as we know from integrals of real functions. FTC also applies; if the complex-valued function of a real variable W is found s.t.

$$U'(t) = u(t),\ V'(t) = v(t)$$

(i.e., if W is the antiderivative of w), then

$$\int_a^b w(t)\,dt = W(b) - W(a)$$

4.2 Notes

- The mean value theorem for derivatives or integrals doesn't always apply, due to the wrapping (periodicity) of the complex plane.

5 Contours, contour integrals, and the ML-inequality

5.1 Definitions

An arc is the set of points generated by a continuous parameterization of x and y on the complex plane over the same interval. In other words, $z = z(t)$ $(a \leq t \leq b)$ is an arc if

$$x = x(t),\ \ y = y(t)$$

are piecewise continuous over the interval $[a, b]$. An arc is simple if it does not self-intersect. An arc is called a simple closed curve if it is simple except for the fact that its end and starts points are the same. It is positively oriented if the interior of the loop is always on the left (i.e., traveling CCW). A differentiable arc is one in which $z'(t)$ is continuous, where

$$z'(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$$

An arc is smooth if its derivative is continuous and its value is nonzero in the open interval (a, b).

A contour is a piecewise smooth arc (i.e., it is composed of a finite number of smooth arcs joined end to end). A simple closed countour (SCC) is a contour where only the final and initial points are the same. Like in the real case, the contour (line) integral is defined as follows:

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt$$

If $|f(z)| \leq M$ $\forall z \in C$, then the upper bound on the modulus of the integral is

$$\left|\int_C f(z)\,dz\right| \leq ML$$

where L is the arc length (shown below). A slightly more general case is that

$$\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$$

5.2 Notes

- The same arc can be represented by multiple parameterizations, and there may be different arcs with the same set of points (i.e., if an arc overlaps itself)

- If we change the parameterization of an arc, we still get the same value when integrating some function over it. I.e., no matter the parameterization of an arc,

$$L = \int_a^b |z'(t)|\, dt$$

is the arc length and is invariant of the parameterization of the arc.

- The contour integral may begin or end on a branch cut of the integrand.

6 Major integral theorems

6.1 Definitions

A simply connected domain is one s.t. every SCC completely inside the domain only encloses points in the domain. (Anything else is a multiply-connected domain.)

6.2 Antiderivative theorem

Suppose that a function $f(z)$ is continuous in a domain D. TFAE:

1. $f(z)$ has an antiderivative $F(z)$ throughout D.
2. Integrals of $f(z)$ along contours lying entirely in D have the same value (path independence):

$$\int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1)$$

3. For a CC lying completely in D,

$$\oint_C f(z)\, dz = 0$$

Note that the antiderivative of a function is unique (if it exists) and is necessarily differentiable.

6.3 Cauchy-Goursat theorem (CG)

If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\oint_C f(z)\, dz = 0$$

6.4 Principle of deformation of paths (PDP)

Let C_1 and C_2 denote POSCCs s.t. C_1 is interior to C_2. If a function f is analytic on and between them, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

6.5 Cauchy integral formula (CIF)

Let f be analytic AOIC a SCC C. If z_0 is a point interior to C, then:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

We denote the use of this formula with argument n to be CIF(n).

6.6 Summary of major conditions and results

Conditions:

1. Known antiderivative throughout some domain. (Doesn't have to be simply connected.)
2. A closed curve in that domain. (Doesn't have to be simple.)

then integrals around a closed loop evaluate to zero, and two integrals to same place evaluate to the same value.

Conditions:

1. SCC C
2. f AOIC C

then the integral around the curve evaluates to zero.

Conditions:

1. POSCC C_2
2. POSCC C_1 completely contained within C_2
3. f analytic on and between C_1 and C_2

then the integrals of f around C_1 and C_2 are equal.

Conditions:

1. POSCC C
2. f AOIC C

3. Integral of form

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

where z_0 lies in the interior of C, and $n \geq 0 \in \mathbb{Z}$

then the integral evaluates to

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

7 Other theorems (not for calculating integrals)

7.1 Theorems for simply-connected domains

- If a function is analytic throughout a domain D, then

  $$\oint_C f(z)\,dz = 0$$

  for any CC C contained completely within D.

- A function that is analytic throughout a domain D has an antiderivative throughout D.

- Entire functions always possess antiderivatives.

7.2 Consequences of CIF

- If f is analytic at a point, then its derivatives of all orders are analytic there too.

- Let f be continuous on a domain D. If

  $$\oint_C f(z)\,dz = 0$$

  for every CC C in D, then f is analytic throughout D. (This is the converse to a theorem in (Sec. 6.4.).)

- (Cauchy's inequality) Assume f is AOIC a PO circle C_R, centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R, then

  $$\left|f^{(n)}(z_0)\right| \leq \frac{n!M_R}{R^n}$$

7.3 Maximum modulus principle

A secondary result is Gauss's mean value theorem: if f is analytic within and on a given circle centered at z_0 and with radius ρ, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$$

i.e., the value of f at the center is the mean of the values of f along the edge of the circle. The main result is the maximum modulus principle: If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D.

8 Uncovered results

8.1 Liouville's theorem

If a function f is entire and bounded in the complex plane, then f is constant throughout the plane.

8.2 The fundamental theorem of algebra

Any n-th order polynomial

$$P(z) = \sum_{i=0}^{n} a_i z^i$$

has at least one zero. By applying this theorem repeatedly, then an n-th order polynomial has exactly n zeros (not necessarily all distinct). In other words, we can factor any n-th order polynomial $P(z)$ into

$$P(z) = \prod_{i=1}^{n} (z - z_i)$$

where $\{z_i\}$ are the zeros of P, not necessarily all distinct.