MA345 Test 1 Notes

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February 20, 2020

Contents

1 Basic algebraic properties and representations

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$(x + iy)^{-1} = \frac{1}{x^2 + y^2}(x - iy) = \frac{\bar{z}}{|z|^2},\ z \neq 0$$

$$\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2),\ \Im(z_1 + z_2) = \Im(z_1) + \Im(z_2)$$

$$|z|^2 = \Re(z)^2 + \Im(z)^2$$

$$\Re(z) \leq |\Re(z)| \leq |z|,\ \Im(z) \leq |\Im(z)| \leq |z|$$

$$r = |z| = \sqrt{x^2 + y^2},\ \theta = \operatorname{atan2}(y, x)$$

$$|z_1 - z_2| = \text{dist}(z_1, z_2)$$

$$\left|\prod_i z_i\right| = \prod_i |z_i|$$

$$\bar{\bar{z}} = z, \ |\bar{z}| = |z|$$

$$\overline{\prod_i z} = \prod_i \bar{z}, \ \overline{\sum_i z} = \sum_i \bar{z}$$

(Products also work with division, sums also work with subtraction.)

$$\Re(z) = \frac{z + \bar{z}}{2}, \ \Im(z) = \frac{z - \bar{z}}{2i}$$

$$z \in \mathbb{R} \iff z = \bar{z}, \ z \in \mathbb{R} \cup \{z : z = ni, \ n \in \mathbb{R}\} \iff z^2 = \bar{z}^2$$

$$z^2 + \bar{z}^2 = c \text{ is a hyperbola}$$

$$\arg z = \text{Arg} z + 2\pi n, \ n \in \mathbb{Z}$$

$$-\pi < \text{Arg} z \leq \pi$$

$$\arg z_1 z_2 = \arg z_1 + \arg z_2$$

$$\arg z^{-1} = -\arg z$$

$$\overline{e^{i\theta}} = e^{-i\theta}$$

By De Moivre's Theorem:

$$c_k = z^{1/n} = \sqrt[k]{r}\left(\cos\frac{\theta_0 + 2\pi k}{k} + i\sin\frac{\theta_0 + 2\pi k}{k}\right)$$

If $\theta_0 = \text{Arg}\, z$, c_0 is called the principal k-th root of z. For square roots, note that

$$\cos\frac{\theta}{2} = \sqrt{\frac{1 + \cos\theta}{2}}$$

$$(z^{1/m})^{-1} = (z^{-1})^{1/m}$$

1.1 Other notes

- Complex numbers are not ordered; comparisons may only happen between two reals, since $\mathbb{R}$ is a total order.

2 Neighborhoods and regions

- An ε neighborhood of z_0 is the domain $|z - z_0| < \varepsilon$.
- A deleted ε neighborhood of z_0 is the domain $0 < |z - z_0| < \varepsilon$.
- An interior point z_0 of S is one such that there exists some neighborhood of z_0 such that it only contains points in S. E.g., a finite set of points cannot have any interior points.
- An exterior point z_0 of S is one such that there exists no neighborhood of z_0 that only contains points in S. In other words, there exists a neighborhood of z_0 that contains no points in S. In other words, it is a point not in S and not a boundary point in S.
- A boundary point z_0 is one such that for every neighborhood of z_0, there exists points in S and points not in S. E.g., a single isolated point is a boundary point.
- An open set contains none of its boundary points (thus contains only interior points). A closed set contains all of its boundary points. A set's closure is the union of a set and its boundary (thus any closure is closed). If some boundary points contained and some not, then neither open nor closed. If no boundary points, then both open and closed (e.g., $\mathbb{C}$).
- A set is connected if each pair of points may be joined by a polygonal line (a finite collection of line segments joined end-to-end).
- A nonempty open connected set is called a domain.
- A domain with none or any of its boundary points is called a region. (Note: a region cannot be a single point, since there is no empty domain.)
- A set is bounded if there exists $R < \infty$ s.t. every point in S lies in the region $|z| < R$.
- A point is an accumulation point (limit point) if each deleted neighborhood of z_0 contains at least one point in S. S is closed iff it contains all of its accumulation points. E.g., an isolated point is not an accumulation point. E.g., any boundary point that is not a isolated point is a boundary point. E.g., any interior point is an accumulation point. E.g., a finite set of points cannot have any accumulation points.

3 The squaring function

This maps hyperbolas centered at the origin to straight lines. E.g., if $x^2 - y^2 = c_1$, then this gets mapped to the line $u = c_1$. Mappings are not necessarily 1-1.

4 Limits, derivatives, and continuity

$$\lim_{z \to z_0} f(z) = w_0 \iff \exists \delta : |f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

i.e., The limit exists if there exists some deleted δ-neighborhood of the approach point s.t. the image of the entire δ-neighborhood is contained within the ε-neighborhood of the limit. If δ has been found, then any smaller δ may also be used. This definition of a limit guarantees uniqueness and is only applicable to interior points (which is fine because we mostly deal with open sets).

Limits of a complex variable iff limits of real and imaginary parts. (same with continuity)

Continuity works the same as limits, and composition of continuous functions is continuous where they are defined.

Can use different values along two paths for showing limit doesn't exist.

5 Limits at infinity

The ε-neighborhood of infinity is $|z - \infty| < \varepsilon$. $|\frac{1}{z} - 0| < \varepsilon$. Thus:

$$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

$$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$

$$\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(1/z)} = 0$$

A set is unbounded iff every neighborhood of the point at infinity contains at least one point in S.

6 Cauchy-Riemann equations

If a function is differentiable at a point z_0, then the C-R equations are satisfied at that point:

$$u_x = v_y, \ u_y = -v_x$$

and $f'(x) = u_x + iv_x$. The converse is also true if u_x, u_y, v_x, v_y exist throughout a neighborhood of z_0 and are continuous at z_0 (this gives a sufficient condition for differentiability).

In polar coordinates, the equivalent expressions are $ru_r = v_\theta$, $-rv_\theta = u_\theta$, and $f'(z) = e^{-i\theta}(u_r + iv_r)$, with the same sufficient conditions.

In general,

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)$$

By applying the C-R equations, for a differentiable function, this simplifies to

$$\frac{\partial F}{\partial \bar{z}} = 0$$

i.e., a differentiable function is independent of $\bar{z}$.

7 Analytic functions

A function f of the complex variable z is analytic (also: regular, holomorphic) in an open set S if it has a derivative everywhere in the set. It is analytic at a point if it is analytic in some neighborhood of z_0. (Thus analyticity never occurs if analytic only in some isolated point, i.e., not at an accumulation point). If a function is differentiable everywhere in an open set, it is analytic on that set.

If a function is analytic at point z_0, it must also be analytic at each point in a neighborhood of z_0. If we say a $f(z)$ is analytic in a non-open set S, we mean it is analytic in some open set containing S.

Rules of differentiation (e.g., sums, differences, products, quotients, composition, etc.) for differentiable functions on open sets carry over to analytic functions.

If a function is not analytic at a point z_0 but is analytic at some point of every neighborhood of z_0, z_0 is called a singularity.

If a function $f(z)$ and its conjugate $\overline{f(z)}$ are both analytic over a domain D, then $f(z)$ is constant over D (uses C-R to show that $f'(z) = 0$ throughout D); thus, if $f(z)$ is real-valued throughout domain D, then $f(z)$ is constant. Similarly, if $f(z)$ is analytic over D and its modulus is constant over D, then $f(z)$ is constant over D.

8 Harmonic functions

Harmonic functions satisfy Laplace's equation, i.e., $H_{xx} + H_{yy} = 0$. The real and imaginary components of a function over a domain which it is analytic satisfy Laplace's equation.

The families of level curves of the component functions of an analytic function are orthogonal if $f'(z_0) \neq 0$.

The polar form of Laplace's equation holds for the component functions when given an input w.r.t. r, θ.

$$r^2 u_{rr} + r u_r r, \theta + u_{\theta\theta} = 0$$

9 Misc. Theorems

Theorem 1 (Triangle inequality)

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

(This can be extended to higher orders by induction.)

Theorem 2 *For any polynomial, $\exists R$ s.t.*

$$\left|\frac{1}{P(z)}\right| \leq \frac{2}{|a_n|R^n}$$

whenever $z > R$.

Theorem 3 (De Moivre's Theorem)

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Theorem 4

$$|z_1| = |z_2| \iff \exists c_1, c_2 \in \mathbb{C} : z_1 = c_1 c_2,\ z_2 = c_1 \bar{c_2}$$

(For the forward direction, let $c_1 = \sqrt{r}\exp(\frac{i}{2}(\theta_1+\theta_2))$, $c_2 = \sqrt{r}\exp(\frac{i}{2}(\theta_1-\theta_2))$).

Theorem 5

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

Theorem 6 (Lagrange's trigonometric identity)

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}$$