ECE211 – Pset 7

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1 Discrete time

Consider a discrete-time state-space realization $\{A, B, C, D\}$ and corresponding transfer function matrix $H(z)$. Let $x[n]$, $u[n]$, $y[n]$ denote the state vector, input vector, and output vector, respectively. Assume N state variables, m inputs, and n outputs.

1. *Write the state-space equations.*

$$x[n+1] = Ax[n] + Bu[n]$$

$$y[n] = Cx[n] + Du[n]$$

2. *Write the formula for $H(z)$ in terms of A, B, C, D.*

$$H(z) = C(zI - A)^{-1}B + D$$

3. *Give the dimensions of A, B, C, D, H.*

$$A : N \times N$$

$$B : N \times m$$

$$C : n \times N$$

$$D : n \times m$$

$$H : n \times m$$

4. *Under certain conditions, the state evolves in the form $x[n] = \Phi[n]x[0]$.*

   (a) *Specify those conditions.*
   The input is zero. (i.e., $u[n] = 0$), $n > 0$, and the system is LTI.

   (b) *State the name of $\Phi[n]$.*
   It is called the state transition matrix.

   (c) *Express $\Phi[n]$ in terms of A, B, C, D.*

$$\Phi[n] = A^n$$

(d) *State the one-sided z-transform of* $\Phi[n]$*, in terms of* A*,* B*,* C*,* D*.*

$$z(zI - A)^{-1}$$

(e) *Give precise conditions on* A*,* B*,* C*,* D *for which* $\Phi[n] \Rightarrow 0$ *as* $n \Rightarrow \infty$*.*
All eigenvalues of A are in the stability region ($|\lambda| < 1$).

2 Continuous time

1. *Write the state-space equations*

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

2. *Write the formula for* $H(s)$*.*

$$H(s) = C(sI - A)^{-1}B + D$$

3. *Give the dimensions of the matrices.*

   Same as for the discrete-time case.

4. *Under certain conditions, the state evolves in the form* $x(n) = \Phi(n)x(0)$*.*

   (a) *Specify those conditions.*
   The input is zero, (i.e., $u(n) = 0$), $t > 0$, and the system is LTI.

   (b) *State the name of* $\Phi(n)$*.*
   It is called the state transition matrix.

   (c) *Express* $\Phi(n)$ *in terms of* A*,* B*,* C*,* D*.*

$$\Phi(n) = e^{At}$$

   (d) *State the one-sided Laplace transform of* $\Phi(n)$*, in terms of* A*,* B*,* C*,* D*.*

$$(sI - A)^{-1}$$

   (e) *Give precise conditions on* A*,* B*,* C*,* D *for which* $\Phi(n) \Rightarrow 0$ *as* $n \Rightarrow \infty$*.*
   All eigenvalues of A are in the stability region ($\Re[\lambda] < 0$).

3 Example transfer function

Given the following transfer function matrix of a continuous-time system:

$$H(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)^2(s+3)} & 0 \\ \frac{1}{s+3} & \frac{s+4}{(s+2)(s+3)} & \frac{1}{s+4} \end{bmatrix}$$

Although not specified, assume underlying state-space realization $\{A, B, C, D\}$.

1. *Specify the transfer function from the first input to the second output.*

$$H(s) = \frac{1}{s+3}$$

2. *List the system poles with multiplicity*

$$p = -2, m = 2$$

$$p = -3, m = 1$$

$$p = -4, m = 1$$

3. *What information is known about the eigenvalues of* A*?*

   A must have the eigenvalue -2 (with multiplicity ≥ 1), eigenvalue -3 (with multiplicity ≥ 2), and eigenvalue -4 (with multiplicity ≥ 1). Their multiplicities may be higher because of pole-zero cancellation.

4. *What is the minimum size of* A*?*

   4×4

5. *Suppose there is a hidden pole at* 4*. How does that change the answers to parts (3) and (4), if all?*

   It means that A must also have an eigenvalue at 4 (with multiplicity ≥ 1), and that the minimum size of A is 5×5.

6. *Again assuming there is a hidden pole at* 4*. Is the system internally stable, and is the system externally stable?*

   Internally unstable (because of pole in A in the RHP), but externally stable (because no pole in the RHP in $H(z)$, input cancels out pole of internal state).

7. *Assume* H *is as given, there is a hidden pole at* 4*, and there are no other hidden poles (including no higher multiplicities other than as system poles). We are now interested in* e^{At}.

   (a) *What is the size of* A*?*
   5×5

(b) *We can compute e^{At} by evaluating a polynomial of the form $r_0(t)I + r_1(t)A + \cdots + r_M(t)A^M$. What is M?*

By the Cayley-Hamilton theorem, we can evaluate a polynomial of degree $M = 5 - 1 = 4$ to evaluate e^{At}.

(c) *Write the general formula for an entry of e^{At}.*

Let $\{\lambda_k\} = \{-2, -3, -4, 4\}$ denote the eigenvalues of A, $\{m_k\} = \{2, 1, 1, 1\}$ denote the multiplicities of the eigenvalues, and $L = 4$ denote distinct eigenvalue count. Then

$$\left(e^{At}\right)_{ij} = \sum_{k=1}^{L} p_k(t)e^{\lambda_i t} = (a_0 + a_1 t)e^{-2t} + a_2 e^{-3t} + a_4 e^{-4t} + a_5 e^{4t}$$

where $p_k(t)$ is a polynomial of degree at most $m_k - 1$.