ECE211 – Pset 5

Jonathan Lam

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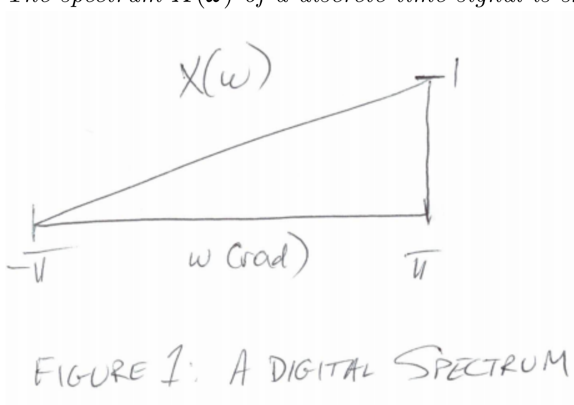
1. *The spectrum $X(\omega)$ of a discrete time signal is shown below.*

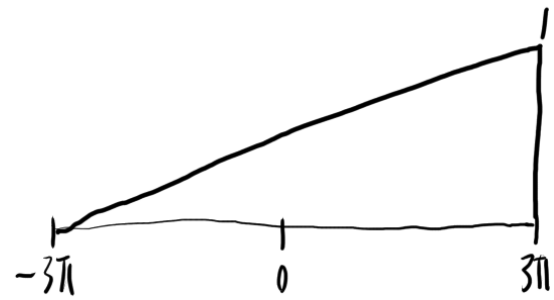


   (a) *Sketch $X(3\omega)$ on the range $-\pi < \omega < \pi$. Briefly state why this is a valid spectrum for some discrete-time signal.*

   This spectrum is valid because it is contained within a 2π interval; that means that there are no "disagreements" for any frequencies that are $2\pi k$ apart. In other words, we are only looking at one branch of $e^{i\theta}$, so this is fine.

   (b) *Sketch $X(4\omega)$ on the range $-\pi < \omega < \pi$.*

(c) *Sketch $X(\omega/3)$ on the range $-3\pi < \omega < 3\pi$. Briefly state why this is not a valid spectrum for some discrete-time signal.*



The spectrum of a signal must be 2π-periodic, since $e^{i\theta}$ is 2π-periodic; thus this spectrum cannot belong to any signal. (E.g., the frequencies at 0 and 2π should match if it were a valid spectrum.)

(d) *Let $Y(\omega) = X(\omega/3)$. Note that $X(\omega/3 - 2\pi k/3)$ ($k \in \mathbb{Z}$) is not obtained by shifting $Y(\omega)$ by $2\pi k/3$. How much is Y shifted by?*

By the definition of Y:

$$X\left(\frac{\omega - 2\pi k}{3}\right) = Y\left(3\left(\frac{\omega - 2\pi k}{3}\right)\right) = Y(\omega - 2\pi k)$$

Thus Y is shifted by $2\pi k$.

2. *Let $y[n] = x[Mn]$. This is called decimation by M. Here $M \geq 2 \in \mathbb{Z}$.. It turns out that*

$$Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega - 2\pi k}{M}\right)$$

Confirm that this produces a valid DTFT for some discrete-time signal.

If X is a valid spectrum, then it is 2π-periodic. Thus $X(\omega/M - 2\pi k/M)$ is $2\pi/M$-periodic, which is also 2π-periodic. Thus Y is the sum of M 2π-periodic spectra and scaling by $1/M$, and thus it should also be 2π-periodic and thus a valid spectrum for some signal.

3. *Let $x[n]$ have DTFT $X(\omega)$. Find the Fourier transform of $x^*[n_0 - n]$.*

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\text{DTFT}\left[x^*[n_0 - n]\right] = \sum_{n=-\infty}^{\infty} x^*[n_0 - n]e^{-j\omega n} = \cdots$$

Let $n' = n_0 - n$ ($n = n_0 - n'$). We have a one-to-one correspondence between n and n', so we can make a change of indices.

$$\cdots = \sum_{n=n_0-n'=-\infty}^{\infty} x^*[n']e^{-j\omega(n_0-n')} = e^{-j\omega n_0}\left[\sum_{n'=-\infty}^{\infty} x^*[n']\left(e^{-j\omega n'}\right)^*\right]$$

$$= e^{-j\omega n_0}\left[\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right]^* = e^{-j\omega n_0}X^*(\omega)$$

(In other words, this is a reversal, shift, and conjugation in time, resulting in conjugation and multiplication by a constant in the transform domain.)

4. *Let $x(t)$ have CTFT $X(\omega)$. Find the inverse Fourier transform of $X^*(\omega_0 - \omega)$.*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$\text{CTFT}^{-1}\left[X^*(\omega_0 - \omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega_0 - \omega)e^{j\omega t}\,d\omega = \cdots$$

Make the substitution $\omega' = \omega_0 - \omega$ ($\omega = \omega_0 - \omega'$, $d\omega = -d\omega'$). We have a one-to-one correspondence, so we can make a change of variable from ω to ω'.

$$\cdots = \frac{1}{2\pi}\int_{\infty}^{-\infty} X^*(\omega')e^{j(\omega_0-\omega')t}(-1)d\omega' = e^{j\omega_0 t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega')\left(e^{j\omega' t}\right)^* d\omega'\right]$$

$$= e^{j\omega_0 t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega\right]^* = e^{j\omega_0 t}x^*(t)$$

(In other words, a shift, reversal, and conjugation in the transform domain will result in a conjugation and multiplication in the time domain.)

5. *Define the paraconjugate to be the following in the transform domain:*

$$\tilde{H}(z) = H^*(1/z^*)$$

$$\tilde{H}(s) = H^*(-s^*)$$

(a) *Check that each paraconjugate formula reduces to* $H^*(\omega)$ *in the frequency domain.*

Discrete-time case:

$$H^*(z) = \sum_{n=-\infty}^{\infty} h^*[n] z^n$$

$$H^*(1/z^*) = \sum_{n=-\infty}^{\infty} h^*[n] \left((z^*)^{-1} \right)^n = \sum_{n=-\infty}^{\infty} h^*[n] \left(z^* \right)^{-n}$$

The frequency domain is the circle $z = e^{j\omega}$.

$$H^*(1/z^*) = \sum_{n=-\infty}^{\infty} h^*[n] \left(e^{-j\omega} \right)^{-n} = \left(\sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \right)^* = H^*(\omega)$$

Continuous-time case:

$$H^*(-s^*) = \int_{-\infty}^{\infty} h^*(t) e^{(-s^*)t} \, dt$$

The frequency domain is the line $s = j\omega$.

$$H^*(-s^*) = \int_{-\infty}^{\infty} h^*(t) e^{(-j\omega)^* t} \, dt = \int_{-\infty}^{\infty} h^*(t) e^{j\omega t} \, dt$$

$$= \left(\int_{-\infty}^{\infty} h(t) e^{-j\omega t} \right)^* = H^*(\omega)$$

(b) *Let* $H(s)$*,* $H(z)$ *be second-order systems. Determine* $\tilde{H}$ *in each case.*

Discrete-time case:

$$H(z) = \frac{b_2 z^2 + b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}$$

$$\tilde{H}(z) = H^*(1/z^*) = \left(\frac{b_2 (1/z^*)^2 + b_1 (1/z^*) + b_0}{a_2 (1/z^*)^2 + a_1 (1/z^*) + a_0} \times \left(\frac{z^*}{z^*} \right)^2 \right)^*$$

$$= \frac{(b_2 + b_1 (z^*) + b_0 (z^*)^2)^*}{(a_2 + a_1 (z^*) + a_0 (z^*)^2)^*} = \frac{b_2^* + b_1^* z + b_0^* z^2}{a_2^* + a_1^* z + a_2^* z^2}$$

Continuous-time case:

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

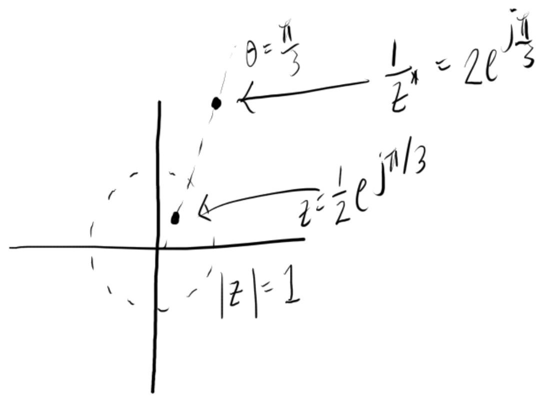
$$\tilde{H}(s) = H^*(-s^*) = \left(\frac{b_2 (-s^*)^2 + b_1 (-s^*) + b_0}{a_2 (-s^*)^2 + a_1 (-s^*) + a_0} \right)^*$$

$$= \frac{(b_2 (s^*)^2 - b_1 (s^*) + b_0)^*}{(a_2 (s^*)^2 - a_1 (s^*) + a_0)^*} = \frac{b_2^* s^2 - b_1^* s + b_0^*}{a_2^* s^2 - a_1^* s + a_0^*}$$

(c) *z and $1/z^*$ are called symmetric points w.r.t. the unit circle, and s and $-s^*$ are called symmetric points w.r.t. the $j\omega$-axis.*

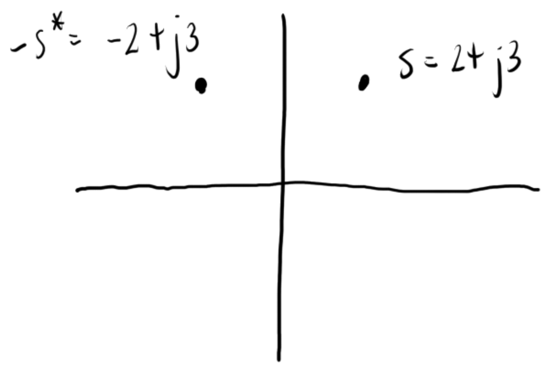
i. *If $z = r\exp(j\theta)$, write $1/z^*$ in polar form.*

$$\frac{1}{z^*} = \frac{1}{r\exp(-j\theta)} = \frac{1}{r}\exp(j\theta)$$



ii. *If $s = \sigma + j\omega$, write $-s^*$ in rectangular form.*

$$-s^* = -(\sigma - j\omega) = -\sigma + j\omega$$



(d) *An all-pass system has constant magnitude 1 at all frequencies. This can be expressed as $|A(\omega)| = 1$. We can also express this as $|A(\omega)|^2 = A(\omega)A^*(\omega) = 1$. In other words, an all-pass function has the following property in the transform domain:*

$$A\tilde{A} = 1$$

In general, the symmetric point to a pole will be a zero, and the most general form for a digital all-pass function is

$$A(z) = z^L \prod_{i=1}^{M} \frac{\alpha_i^* z - 1}{z - \alpha_i}$$

where $|\alpha_i| \neq 0$ *and* $L \in \mathbb{Z}$.

i. *Plug in* $z = \exp(j\omega)$ *and confirm* $A(z)$ *is an all-pass function.*

Since $|z| = 1$, $z = 1/z^*$. Thus $\tilde{A}(z) = A^*(1/z^*) = A^*(z)$, and $A\tilde{A} = AA^* = |A|^2$. Thus we have to show that $|A|^2 = 1$.

$$|A(z)| = \left| \left(e^{j\omega}\right)^L \prod_{i=1}^{M} \frac{\alpha_i^* e^{j\omega} - 1}{e^{j\omega} - \alpha_i} \right| = \left|e^{j\omega L}\right| \prod_{i=1}^{M} \left| \frac{\alpha_i^* e^{j\omega} - 1}{e^{j\omega} - \alpha_i} \right|$$

It is easy to see that $\left|e^{j\omega L}\right| = \sqrt{e^{j\omega L} e^{-j\omega L}} = 1$. The multiplicand (the magnitude inside the product operator) also has value 1:

$$\left| \frac{\alpha_i^* e^{j\omega} - 1}{e^{j\omega} - \alpha_i} \right| = \left| \frac{-e^{j\omega}\left(e^{-j\omega} - \alpha_i^*\right)}{e^{j\omega} - \alpha_i} \right| = \left|-e^{j\omega}\right| \left| \frac{\left(e^{j\omega} - \alpha_i\right)^*}{e^{j\omega} - \alpha_i} \right|$$

Since $|(-)e^{j\omega}| = 1$ and $|\beta^*/\beta| = 1 \, \forall \beta \in \mathbb{C}$, this magnitude has value 1. Thus:

$$|A(z)| = (1) \prod_{i=1}^{M} (1) = 1 \Rightarrow |A|^2 = A\tilde{A} = 1$$

ii. *If* $L \neq 0$, *the factor* z^L *introduces a set of pole-zero pairs. Assume* $L > 0$. *What are the poles and zeros for this factor?*

This introduces a set of poles at infinity and zeros at zero, with multiplicity L. This obeys the symmetry of the poles, since $|0| = |1/\infty|$.

6. *Use the method of partial fractions to find the (causal) inverse Laplace transform for each of the following.*

(a) *Basic case:*

$$H(s) = \frac{3s^3 + 2}{(s+2)(s+3)}$$

$$H(s) = As + B + \frac{C}{s+2} + \frac{D}{s+3}$$

$$\begin{aligned} As + B \quad & = 3s - 15 \\ C = \frac{3(-2)^3 + 2}{-2 + 3} \quad & = -22 \\ D = \frac{3(-3)^3 + 2}{-3 + 2} \quad & = 79 \end{aligned}$$

$$H(s) = 3s - 15 - \frac{22}{s+2} + \frac{79}{s+3}$$

$$\mathcal{L}^{-1}\{H(s)\} = h(t) = 3\delta'(t) - 15\delta(t) - 22e^{-2t}u(t) + 79e^{-3t}u(t)$$

(b) *Pole with multiplicity:*

$$H(s) = \frac{3s^3 + 2}{(s+2)(s+3)^2}$$

$$H(s) = A + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$\begin{aligned}
A & & &= 3 \\
B &= \frac{3(-2)^3 + 2}{(-2+3)^2} & &= -22 \\
C &= \frac{d}{ds}\left(\frac{3s^3+2}{s+2}\right)\bigg|_{s=-3} = \frac{6(-3)^3 + 18(-3)^2 - 2}{(-3+2)^2} & &= -2 \\
D &= \frac{3(-3)^3 + 2}{(-3+2)} & &= 79
\end{aligned}$$

$$H(s) = 3 - \frac{22}{s+2} - \frac{2}{s+3} + \frac{79}{(s+3)^2}$$

$$\mathcal{L}^{-1}\left\{H(s)\right\} = h(t) = 3\delta(t) - 22e^{-2t}u(t) - 2e^{-3t}u(t) + 79te^{-3t}u(t)$$

(c) *Complex poles:*

$$H(s) = \frac{3s^3 + 2}{(s+3)(s^2 + 4s + 8)}$$

$$H(s) = A + \frac{B}{s+3} + \frac{C}{s-(-2-2j)} + \frac{D}{s-(-2+2j)}$$

$$\begin{aligned}
A & & &= 3 \\
B &= \frac{3(-3)^3 + 2}{(-3)^2 + 4(-3) + 8} & &= \frac{-79}{5} \\
C &= \frac{3\left((2)^{3/2}\, e^{-j3\pi/4}\right)^3 + 2}{((-2-2j)+3)((-2-2j)+2-2j)} \\
&= \frac{3(16-16j)+2}{(1-2j)(-4j)} = \frac{25-24j}{-4-2j} & &= -\frac{13}{5} + \frac{73}{10}j \\
D &= \frac{3\left((2)^{3/2}\, e^{j3\pi/4}\right)^3 + 2}{((-2+2j)+3)((-2+2j)+2+2j)} \\
&= \frac{3(16+16j)+2}{(1+2j)(4j)} = \frac{25+24j}{-4+2j} & &= -\frac{13}{5} - \frac{73}{10}j
\end{aligned}$$

$$H(s) = 3 - \frac{79/5}{s+3} + \frac{-13/5 + 73/10j}{s-(-2-2j)} + \frac{-13/5 - 73/10j}{s-(-2+2j)}$$

$$\mathcal{L}^{-1}\left\{H(s)\right\} = h(t)$$

$$= 3\delta(t) - \frac{79}{5}e^{-3t}u(t) + \left(-\frac{13}{5} + \frac{73}{10}j\right)e^{(-2-2j)t}u(t) + \left(-\frac{13}{5} - \frac{73}{10}j\right)e^{(-2+2j)t}u(t)$$

$$= 3\delta(t) - \frac{79}{5}e^{-3t}u(t) + e^{-2t}\left[\left(-\frac{13}{5} + \frac{73}{10}j\right)e^{-2jt} + \left(-\frac{13}{5} - \frac{73}{10}j\right)e^{2jt}\right]u(t)$$

Let

$$z = \left(-\frac{13}{5} + \frac{73}{10}j\right)e^{-2jt} = \left(-\frac{13}{5} + \frac{73}{10}j\right)(\cos(2t) - i\sin(2t))$$

Then the part in the square brackets is

$$z + z^* = 2\,\mathrm{Re}(z) = 2\left(-\frac{13}{5}\cos(2t) - \frac{73}{10}(-\sin(2t))\right)$$

and

$$h(t) = 3\delta(t) - \frac{79}{5}e^{-3t}u(t) + e^{-2t}\left(-\frac{26}{5}\cos(2t) + \frac{73}{5}\sin(2t)\right)u(t)$$

7. *Compute the integral:*

$$\int_0^5 7e^{-2t}[\delta(t-1) + 2\delta(t-10) + 5\delta'(t-2)]\,dt$$

$$= 7\int_0^5 e^{-2t}\delta(t-1)\,dt + 14\int_0^5 e^{-2t}\delta(t-10)\,dt + 35\int_0^5 e^{-2t}\delta'(t-2)\,dt$$

$$= 7\left. e^{-2t}\right|_{t=1} + 14(0) + 35\left.\left[(-1)^1\frac{d}{dt}e^{-2t}\right]\right|_{t=2}$$

$$= 7e^{-2} - 35(-2)e^{-2(2)} = 7e^{-2} + 70e^{-4}$$

8. *Windowing*

```
clc; clear all; clear screen;

% Pset 5 Question 8
% Jonathan Lam

% 8c
N = 15;
r = 30;
beta = 3.05;

hr = rectwin(N);          % rectangular window
hc = chebwin(N, r);       % Chebyshev window
hk = kaiser(N, beta);     % Kaiser window
```

```
w = linspace(0, pi, 1000);

% normalization
hr = hr/sum(hr);
hc = hc/sum(hc);
hk = hk/sum(hk);

H = freqz(hr, 1, w);
plot(w, 20*log10(abs(H)));
hold on;

H = freqz(hc, 1, w);
plot(w, 20*log10(abs(H)));
hold on;

H = freqz(hk, 1, w);
plot(w, 20*log10(abs(H)));
ylim([-50 0]);
xlim([0 pi]);
ylabel(["Magnitude" "(dB)"]);
xticks(0:pi/4:pi);
xticklabels(["0","\pi/4","\pi/2","3\pi/4","\pi"]);
xlabel(["Digital frequency" "(rad)"]);
legend(["Rectangular" "Chebyshev", "Kaiser"]);
title("Magnitude response of window functions");

% 8c
figure;

subplot(1, 3, 1);
stem(hr);
ylim([0 0.12]);
xlim([0 16]);
ylabel("h[n]");
xlabel("n");
title("Rectangular window");

subplot(1, 3, 2);
stem(hc);
ylim([0 0.12])
xlim([0 16]);
ylabel("h[n]");
xlabel("n");
title("Chebyshev window");

```

```
subplot(1, 3, 3);
stem(hk);
ylim([0 0.12]);
xlim([0 16]);
ylabel("h[n]");
xlabel("n");
title("Kaiser window");

% 8d
figure;

H = freqz(hr, 1, w);
plot(w, unwrap(180/pi*angle(H)));
ylabel(["Phase" "(deg)"]);
xlabel(["Digital frequency" "(rad)"]);
xticks(0:pi/4:pi);
xticklabels(["0","\pi/4","\pi/2","3\pi/4","\pi"]);
title("Phase response of rectangular window");
```

(a) *Why does $h = h/sum(h)$ yield $H(0) = 1$? (This assumes $sum(h) \neq 0$; if it is zero, what is $H(0)$?)*

Normalizing h makes the sum of the coefficients equal to one:

$$\sum_{-\infty}^{\infty} h[n] = 1$$

Since

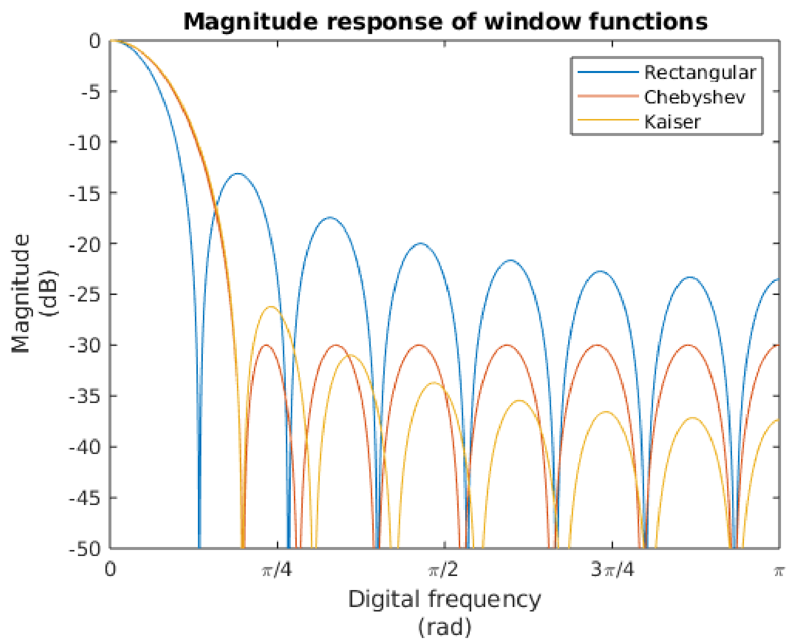
$$H(\omega) = \sum_{-\infty}^{\infty} h[n]e^{-j\omega n}$$

then

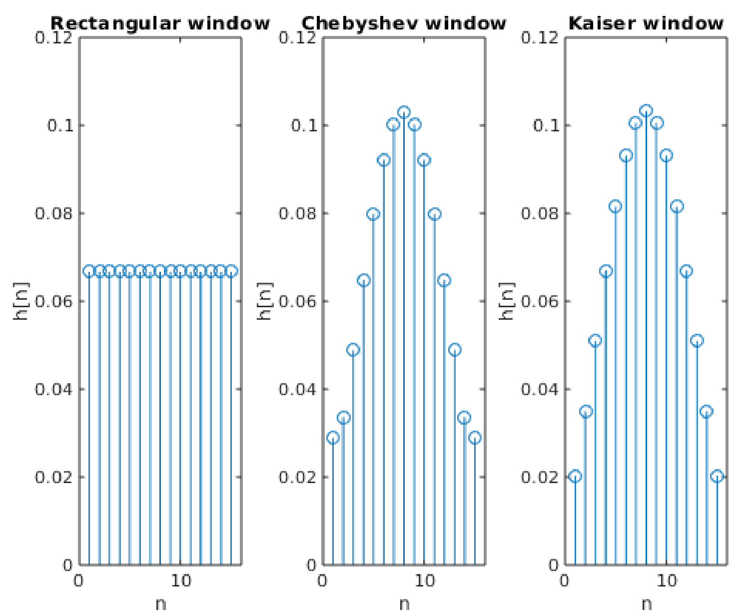
$$H(0) = \sum_{-\infty}^{\infty} h[n]e^{0} = \sum_{-\infty}^{\infty} h[n] = 1$$

If $\text{sum}(h) = 0$, then we wouldn't be able to normalize it in the first place. But if we wanted to calculate $H(0)$, it would be equal to $\text{sum}(h) = 0$ by the same reasoning as above.

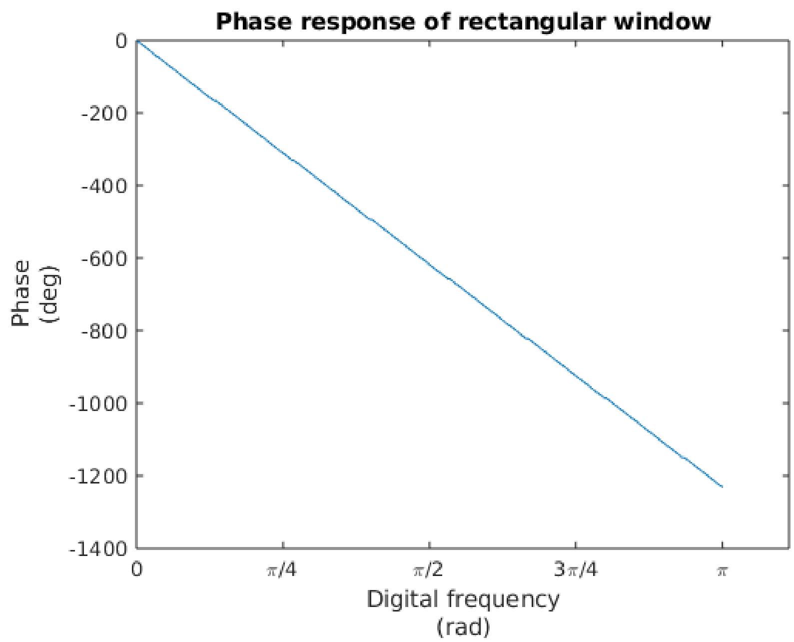
(b) Figure:



(c) Figure:



(d) Figure:

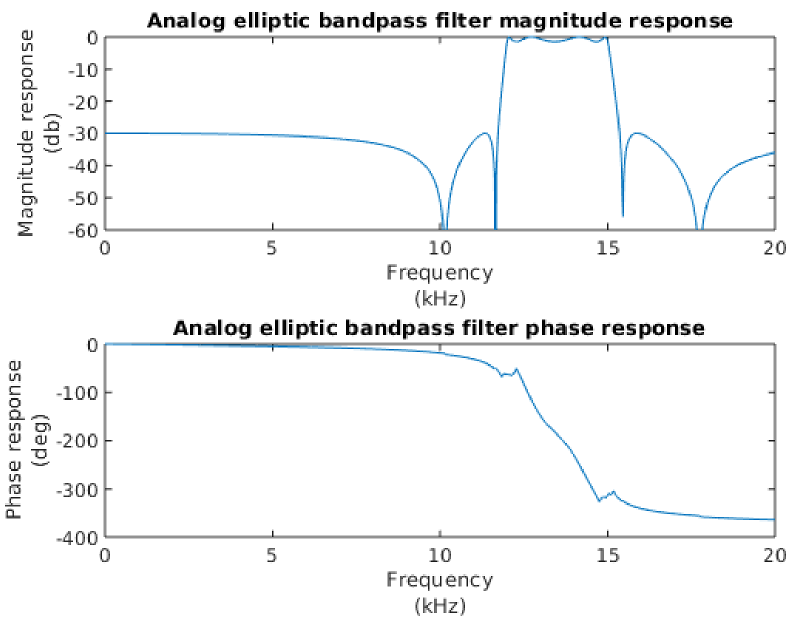


9. *Analog elliptic bandpass filter*

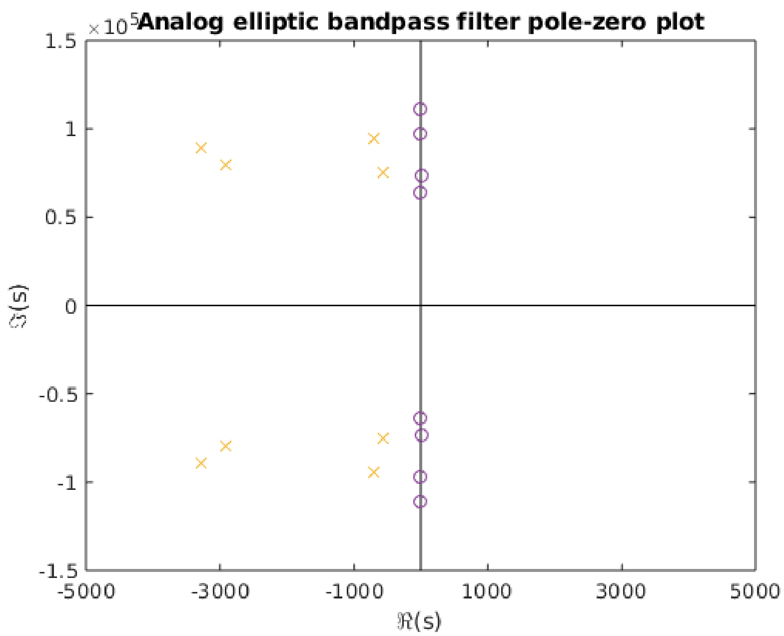
```
% Pset 5 Question 9
% Jonathan Lam
clc; clear all;

% design a bandpass analog elliptic filter
fp = [12e3 15e3];
fs = [10e3 16e3];
rp = 1.5;
rs = 30;
[n, wn] = ellipord(2*pi*fp, 2*pi*fs, rp, rs, 's');
[z, p, k] = ellip(n, rp, rs, wn, 's');
[b, a] = zp2tf(z, p, k);

% 9a
w = linspace(0, 2*pi*20e3, 1e3);
H = freqs(b, a, w);

% 9b
figure;
subplot(2, 1, 1);
plot(w, 20*log10(abs(H)));
ylim([-60 0]);
xlim([0 2*pi*20e3]);
```

```
yticks(-60:10:0);
xticks(0:2*pi*5e3:2*pi*20e3);
xticklabels(["0" "5" "10" "15" "20"]);
ylabel(["Magnitude response" "(db)"]);
xlabel(["Frequency" "(kHz)"]);
title("Analog elliptic bandpass filter magnitude response");

subplot(2, 1, 2);
plot(w, unwrap(180/pi*angle(H)));
xlim([0 2*pi*20e3]);
xticks(0:2*pi*5e3:2*pi*20e3);
xticklabels(["0" "5" "10" "15" "20"]);
ylabel(["Phase response" "(deg)"]);
xlabel(["Frequency" "(kHz)"]);
title("Analog elliptic bandpass filter phase response");

% 9c
figure;
plot([-1e7i 1e7i], 'k');     % plot imaginary axis
hold on;
plot([-1e4 1e4+0.1i], 'k'); % plot real axis
hold on;
plot(p, 'x');                % plot poles
hold on;
plot(z, 'o');                % plot zeroes
ylim([-1.5e5 1.5e5]);
xlim([-5e3 5e3]);
yticks(-1.5e5:0.5e5:1.5e5);
xticks(-5e3:2e3:5e3);
ylabel('\Im(s)');
xlabel('\Re(s)');
title("Analog elliptic bandpass filter pole-zero plot");
```

(a) Figure:



(b) Figure:



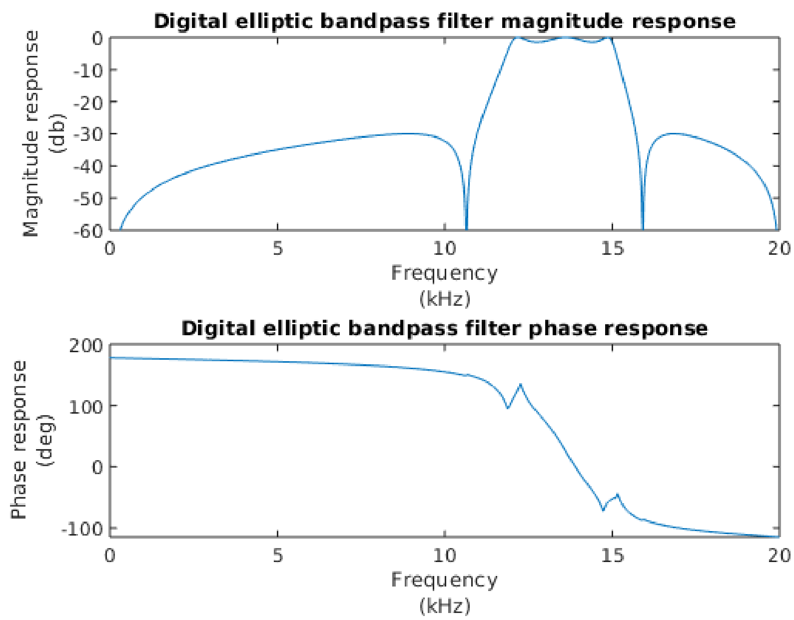
10. *Digital elliptic bandpass filter*

```
clc; clear all;
```

```
% Pset 5 Question 10
% Jonathan Lam

% design digital filter with same specs as q9
fp = [12e3 15e3];
fs = [10e3 16e3];
rp = 1.5;
rs = 30;
fsamp = 40e3;
Bnyq = fsamp/2;
fpd = fp/Bnyq;
fsd = fs/Bnyq;
[nd, fnd] = ellipord(fpd, fsd, rp, rs); % freq specs normalized to Nyq BW
[zd, pd, kd] = ellip(nd, rp, rs, fnd);
[bd, ad] = zp2tf(zd, pd, kd);

% compute frequency response
w = linspace(0, 2*pi*20e3, 1e3);
wd = w/fsamp;
H = freqz(bd, ad, wd);

% plot magnitude and phase response
figure;
subplot(2, 1, 1);
plot(wd, 20*log10(abs(H)));
ylim([-60 0]);
xlim([0 pi]);
yticks(-60:10:0);
xticks(0:pi/4:pi);
xticklabels((0:pi/4:pi)/2/pi*fsamp/1e3);
ylabel(["Magnitude response" "(db)"]);
xlabel(["Frequency" "(kHz)"]);
title("Digital elliptic bandpass filter magnitude response");

subplot(2, 1, 2);
plot(wd, unwrap(180/pi*angle(H)));
xlim([0 pi]);
xticks(0:pi/4:pi);
xticklabels((0:pi/4:pi)/2/pi*fsamp/1e3);
ylabel(["Phase response" "(deg)"]);
xlabel(["Frequency" "(kHz)"]);
title("Digital elliptic bandpass filter phase response");

% plot pole-zero plot
figure;
zplane(zd, pd);
```

```
title("Digital elliptic bandpass filter pole-zero plot");
```

(a) Figure:



(b) Figure: