PSET 4: Basic z-Transform

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1. *Suppose $H(z)$ is such that it has a zero of multiplicity $m \geq 1$ at ∞. Then it has an inverse transform such that $h[N] \neq 0$ but $h[n] = 0\ \forall n \leq N-1$. Find N.*

   This means $H(z)$ has a single causal region. We focus on this causal region. Multiplying it by z will decrease the multiplicity of the zero at ∞ and shift the corresponding impulse response (its inverse transform) $h[n]$ left once. When the multiplicity of the zero at ∞ becomes negative (i.e., when there is a pole at ∞), then there will be no causal region of $H(z)$ anymore.

   The multiplicity of the zero at ∞ becomes zero when multiplying by z^m, which corresponds to $h[n+m]$. Since $h[n+m]$ is a causal signal and left-shifting it any more would make it non-causal by the above argument, this means that $h[m] \neq 0$ and $h[n] = 0\ \forall n \leq m-1$. Thus $N = m$.

   If we take $m = 0$, that means H has a finite nonzero limit at $z \to \infty$. In this case, what can we conclude about the causal inverse transform $h[n]$?

   With $N = m = 0$, $h[0] \neq 0$, by the above argument.

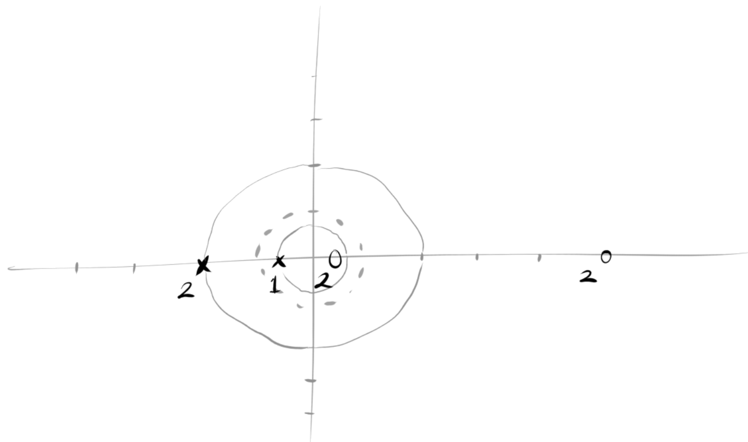
2. *Given:*

   $$H(z) = \frac{3(2z-1)^2(z-5)^2}{(3z+2)(z+2)^2}$$

   (a) *List all poles and zeros, with multiplicities, and draw a pole-zero plot by hand.*

   Poles: -2/3 (m=1), -2 (m=2), ∞ (m=1)
   Zeros: 1/2 (m=2), 5 (m=2)

Figure 1: Pole-zero plot



(b) *Identify all possible ROCs for $H(z)$.*
$|z| < 2/3$, $2/3 < |z| < 2$, $2 < |z|$

(c) *Identify the ROC (if any) associated with:*

- *a causal system*: none; pole at ∞
- *a stable system*: $2/3 < |z| < 2$
- *a system with a well-defined frequency response*: $2/3 < |z| < 2$

(d) *We seek an integer L s.t. $z^L H(z)$ has a causal inverse transform such that $h[0] \neq 0$. Find all possible L.*

This uses much of the same logic as question (1). For there to be a causal inverse transform, there must not be a pole at ∞, so $L < 0$. The operation $z^L H(z)$ implies a left-shift of L on $h[n]$. Since $L < 0$ by the above statement, this means a right-shift by $-L$. Thus, specifying the condition $h[0] \neq 0$ puts a tight bound on L: we must shift $h[z]$ exactly enough units until it is causal, because only then will $h[0] \neq 0$ also be fulfilled.

By this reasoning, L must be -1, since then $z^{-1}H(z)$ is the smallest right-shift that can occur to create a causal region.

3. *A digital FIR filter with input x and output y is given by:*

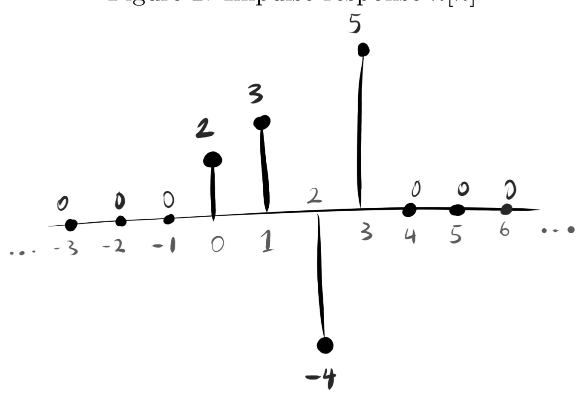
$$y[n] = 2x[n] + 3x[n-1] - 4x[n-2] + 5x[n-3]$$

(*a*) *Sketch the impulse response $h[n]$.* (See figure at top of next page.)

(*b*) *Express $h[n]$ as a superposition of impulses.*

$$h[n] = y[n](\delta) = 2\delta[n] + 3\delta[n-1] - 4\delta[n-2] + 5\delta[n-3]$$

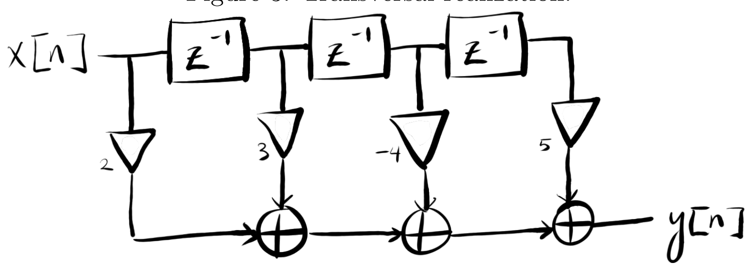
Figure 2: Impulse response $h[n]$



(c) *Specify the length and order of the filter.*
Length: 4 (number of terms)
Order: 3 (highest delay)

(d) *Sketch a transversal filter realization.*

Figure 3: Transversal realization.



4. *Given the following transfer function of a digital IIR filter:*

$$H(z) = \frac{3z^2 + 4z + 5}{10z^2 - z + 2}$$

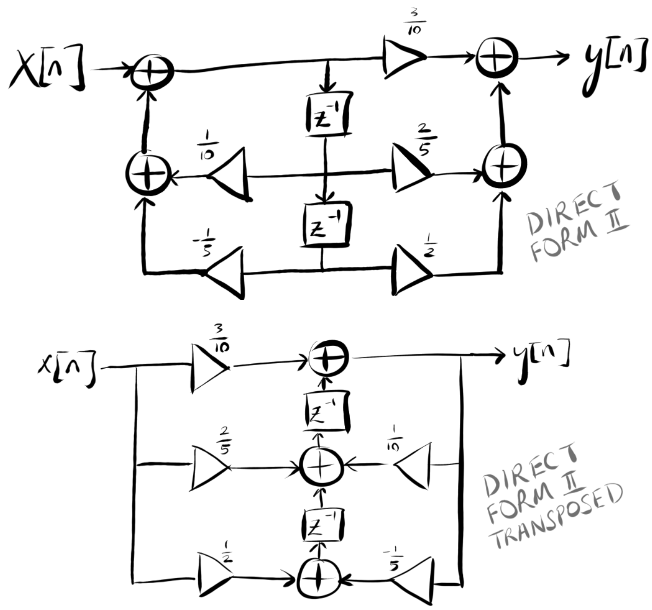
(a) *Write a difference equation with input x and output y.*
Rewriting in nicer form:

$$H(z) = \frac{\frac{3}{10}z^0 + \frac{2}{5}z^{-1} + \frac{1}{2}z^{-2}}{1 - \left(\frac{1}{10}z^{-1} - \frac{1}{5}z^{-2}\right)}$$

$$y[n] = \frac{3}{10}x[n] + \frac{2}{5}x[n-1] + \frac{1}{2}x[n-2] + \frac{1}{10}y[n-1] - \frac{1}{5}y[n-2]$$

(b) *Sketch direct form II and direct form II transposed realizations.*

Figure 4: Direct form II and direct form II transposed realizations.



5. *The following figure shows a block diagram. Find the overall transfer function by inspection.*

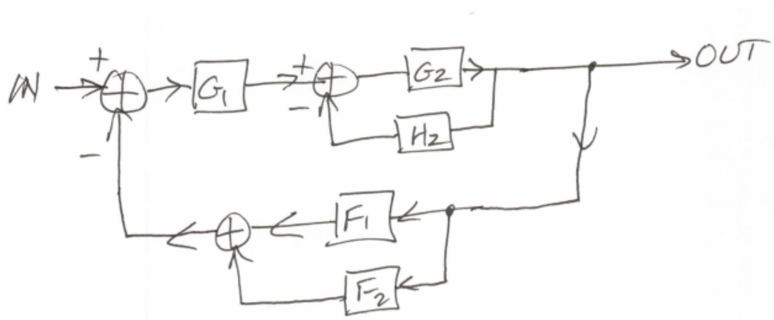


Figure 5: Block diagram of complex transfer function.

$$H = \frac{G_1 \left(\frac{G_2}{1+G_2 H_2}\right)}{1 + G_1 \left(\frac{G_2}{1+G_1 H_2}\right)(F_1 + F_2)}$$