ECE211 – PSET2

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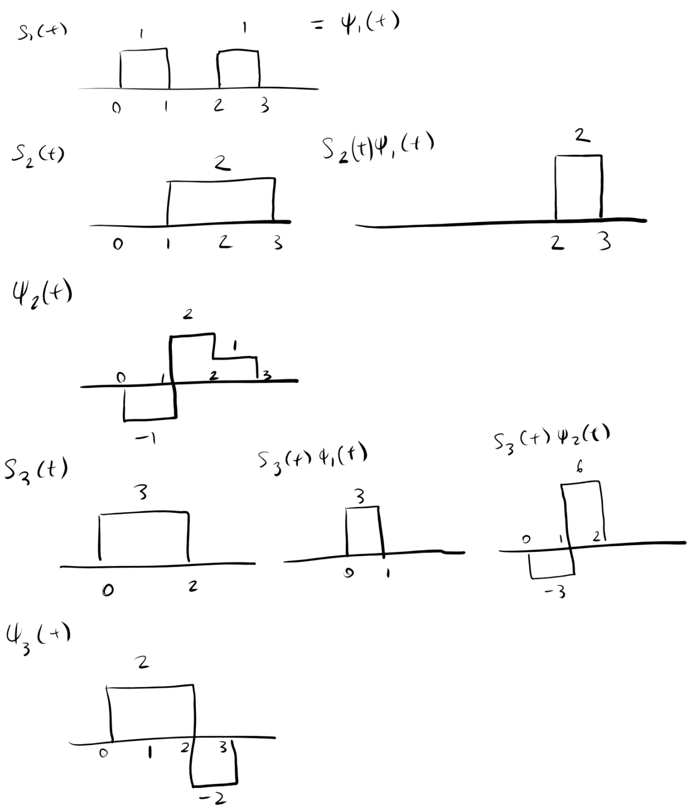
1. Define the inner product $\langle x, y\rangle \coloneqq \int_{-\infty}^{\infty} x(t)y(t)\,dt$ (area under the product of $x(t)$ and $y(t)$).

   (a) *Perform GSO on the three given signals.*

   Sketches: $\{s_i\}$ represent the original signals set; $\{\psi_i\}$ represent an orthogonal basis; and $\{\phi_i\}$ represents the orthonormal basis.



   Calculating orthogonal basis:

$$\psi_1 = s_1$$

$$||\psi_1||^2 = 1^2 + 1^2 = 2$$

$$\langle s_2, \psi_1 \rangle = 2 \cdot 1 = 2$$

$$\psi_2 = s_2 - \frac{\langle s_2, \psi_1 \rangle}{||\psi_1||^2}\psi_1 = s_2 - \frac{2}{2}\psi_1 = s_2 - \psi_1$$

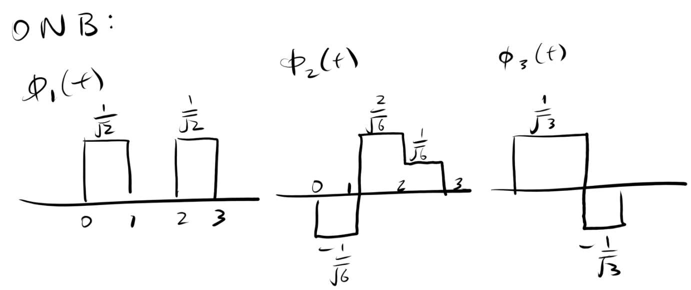
$$||\psi_2||^2 = (-1)^2 + 2^2 + 1^2 = 6$$

$$\langle s_3, \psi_1 \rangle = 3 \cdot 1 = 3$$

$$\langle s_3, \psi_2 \rangle = -3 \cdot 1 + 6 \cdot 1 = 3$$

$$\psi_3 = s_3 - \frac{\langle s_3, \psi_1 \rangle}{||\psi_1||^2}\psi_1 - \frac{\langle s_3, \psi_2 \rangle}{||\psi_2||^2} = s_3 - \frac{3}{2}\psi_1 - \frac{3}{6}\psi_2$$

Normalizing to get orthonormal basis:



$$||\psi_3||^2 = 2^2 + 2^2 + (-2)^2 = 12$$

$$\phi_1 = \frac{1}{||\psi_1||}\psi_1 = \frac{1}{\sqrt{2}}\psi_1$$

$$\phi_2 = \frac{1}{||\psi_2||}\psi_2 = \frac{1}{\sqrt{6}}\psi_2$$

$$\phi_3 = \frac{1}{||\psi_3||}\psi_3 = \frac{1}{\sqrt{12}}\psi_3$$

(b) **Express s_1, s_2, and s_3 w.r.t. the newly-calculated ONB.**

$$s_1 = \langle s_1, \phi_1 \rangle \phi_1 + \langle s_1, \phi_2 \rangle \phi_2 + \langle s_1, \phi_3 \rangle \phi_3 = \sqrt{2} \cdot \phi(1) + 0 \cdot \phi_2 + 0 \cdot \phi_3$$

$$s_2 = \langle s_2, \phi_1 \rangle \phi_1 + \langle s_2, \phi_2 \rangle \phi_2 + \langle s_2, \phi_3 \rangle \phi_3 = \sqrt{2} \cdot \phi(1) + \sqrt{6} \cdot \phi_2 + 0 \cdot \phi_3$$

$$s_3 = \langle s_3, \phi_1 \rangle \phi_1 + \langle s_3, \phi_2 \rangle \phi_2 + \langle s_3, \phi_3 \rangle \phi_3 = \frac{3}{\sqrt{2}} \cdot \phi(1) + \frac{3}{\sqrt{6}} \cdot \phi_2 + \frac{6}{\sqrt{3}} \cdot \phi_3$$

Alternatively, as tuples in $\mathbb{R}^3$:

$$\vec{s}_1 = (\sqrt{2}, 0, 0);\ \vec{s}_2 = (\sqrt{2}, \sqrt{6}, 0);\ \vec{s}_3 = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{6}}, \frac{6}{\sqrt{3}}\right)$$

(c) *Express the projection of s_3 onto span$\{\phi_1, \phi_2\}$ as a vector in $\mathbb{R}^3$.*

$$\hat{s}_3 = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{6}}, 0\right)$$

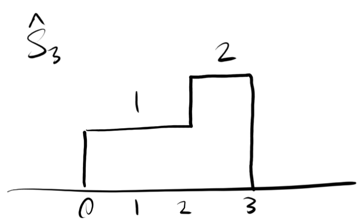
(d) *Compute $||s_3||$ using an integral and compare it to $||\vec{s}_3||$.*

$$||s_3|| = \sqrt{\int_{-\infty}^{\infty} s_3^2(t)\, dt} = \sqrt{3^2 + 3^2 + 0^2} = \sqrt{18}$$

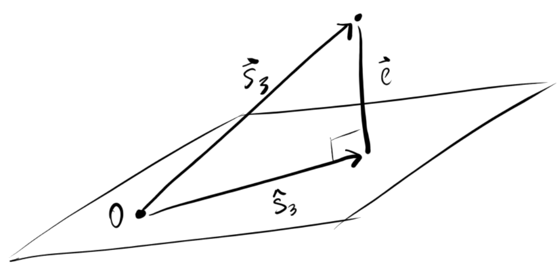
$$||\vec{s}_3|| = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{3}{\sqrt{6}}\right)^2 + \left(\frac{6}{\sqrt{3}}\right)^2} = \sqrt{\frac{9}{2} + \frac{9}{6} + \frac{36}{3}} = \sqrt{18}$$

They match!

(e) *Sketch the approximate signal $\hat{s}_3$.*



(f) *Consider the error equation $e(t) = s_3(t) - \hat{s}_3(t)$. Write the equation that relates $||s_3||$, $||\hat{s}_3||$, and $||e||$ and use it to compute $||e||$.*
(Not really sketching e, just showing the general idea.)



Since $e(t) \perp \hat{s}_3(t)$, and $\hat{s}_3(t) + e(t) = s_3(t)$, the three magnitudes are related by the Pythagorean theorem:

$$||e|| = \sqrt{||s_3||^2 - ||\hat{s}_3||^2} = \sqrt{18 - \left(\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{3}{\sqrt{6}}\right)^2\right)} = \sqrt{18 - 6} = \sqrt{12}$$

2. A voltage amplifier provides a gain of 30dB.

(a) *Given $V_{in} = 1\,V$, find output amplitude.*

$$20 \log_{10} \frac{V_{out}}{1\text{V}} = 30$$

$$V_{out} = (1V)10^{30/20} = 31.6\text{V}$$

(b) *Given $P_{out} = 20dBm$, find input power.* Since the decibel scale is logarithmic, we may simply add the gain to an input voltage on an absolute decibel scale to get the output voltage on the same absolute decibel scale.

$$V_{in}(\text{dBm}) + V_{gain}(\text{dB}) = V_{out}(\text{dBm})$$

$$V_{in} = 20\text{dBm} - 30\text{dB} = -10\text{dBm}$$

(c) Express the input power from part (b) in Watts.

$$-10\text{dBm} = 10 \log_{10} \frac{P_{in}}{1 \times 10^{-3}\text{W}}$$

$$P_{in} = (1 \times 10^{-3}\text{W})10^{-10/10} = 1 \times 10^{-4}\text{W}$$

3. A real 40MHz sine wave is sampled at 100MHz.

(a) *List all frequencies up to 200MHz that would alias into 40MHz.*

40MHz, 60Mhz, 140MHz, 160MHz

(b) *Specify the appropriate cutoff frequency for the filter that should be employed by the 100MHz A/D converter.*

Cutoff frequency should be the Nyquist bandwidth: $f_s/2 = 100\text{MHz}/2 = 50\text{MHz}$ in order to avoid aliasing.

(c) *Specify the input signal frequencies in the different scales.*

analog radian frequency	$80\pi \times 10^6$ rad/s
normalized digital radian frequency	$4\pi/5$ rad
as a fraction of sampling rate	$2/5$
as a fraction of Nyquist bandwidth	$4/5$

(d) *The sampled data is passed to a D/A converter operated at a rate of 60kHz. List the six lowest frequencies that emerge at the output.*

24KHz, 36KHz, 84KHz, 96KHz, 144KHz, 156KHz

(e) *Specify the cutoff frequency of the anti-imaging filter that should be used in the D/A converter.*

Again, the cutoff frequency should be the Nyquist bandwidth: $f_{cutoff} = f_s/2 = 60\text{kHz}/2 = 30\text{kHz}$.

4. *Let $s(t)$ be a bandpass signal with reference carrier frequency f_0 and $s_{BB}(t)$. Derive the correct formula for the baseband equivalent for $s(t-t_0)$.*

$$s(t) = \Re\left(s_{BB}(t)e^{jw_0 t}\right)$$

$$s(t-t_0) = \Re\left(s_{BB}(t-t_0)e^{j2\pi f_0(t-t_0)}\right) = \Re\left(\left(e^{-j2\pi f_0 t_0}s_{BB}(t-t_0)\right)e^{2\pi f_0 t}\right)$$

Thus the baseband signal that corresponds with the shifted bandpass signal is multiplied by $\exp(-j\omega_0 t_0)$ (like shifting in the Laplace transform):

$$s'_{BB}(t) = e^{-j2\pi f_0 t_0}s_{BB}(t-t_0)$$

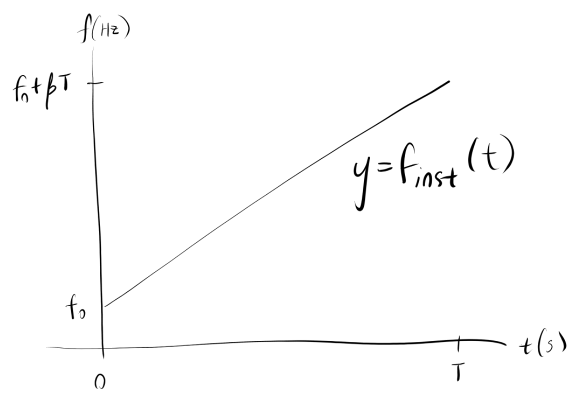
5. A chirp signal is given by

$$s(t) = A\cos(2\pi f_0 t + \pi\beta t^2 + \phi_0),\ \ 0 \le t \le T$$

(a) *Compute and sketch the instantaneous frequency $f_{inst}(t)$, $0 \le t \le T$.*

$$f_{inst}(t) = \frac{1}{2\pi}\frac{d\theta}{dt} = f_0 + \beta t$$

(i.e., frequency is f_0 at time $t = 0$, and increases as time increases with slope β.)



(b) *What would an appropriate unit for β be?*

Since β is the slope in the instantaneous-frequency/time plot, it should be Hz/s (rate of change of frequency).

(c) *Write the baseband equivalent for the chirp.*

$$s(t) = A\cos(2\pi f_0 t + \pi\beta t^2 + \phi_0) = \Re\left(Ae^{j(2\pi f_0 t + \pi\beta t^2 + \phi_0)}\right)$$

$$= \Re\left(Ae^{j(\pi\beta t^2 + \phi_0)}e^{j2\pi f_0 t}\right) \Rightarrow s_{BB}(t) = Ae^{j(\pi\beta t^2 + \phi_0)}$$

(d) *Suppose we receive a delayed version of the chirp: $r(t) = s(t - t_0)$. Write the baseband equivalent.*

$$r_{BB}(t) = e^{-j2\pi f_0 t_0} s_{BB}(t - t_0)$$

(e) *Given $r(t)$ at the receiver the signal is brought to baseband (i.e., $r_{BB}(t)$ is obtained) and then the following operation is performed at baseband:*

$$x_{BB}(t) = r_{BB}(t) s^*_{BB}(t)$$

Obtain the expression $x_{BB}(t)$. Which is the best way to recover the delay t_0?

$$x_{BB}(t) = \left(e^{-j2\pi f_0 t_0} \cdot Ae^{j(\pi\beta(t-t_0)^2+\phi_0)}\right)\left(Ae^{-j(\pi\beta t^2+\phi_0)}\right)$$
$$= A^2 e^{j(\pi\beta t_0^2 - 2\pi(f_0+t\beta)t_0)}$$

This is a complex sine wave with phase

$$\theta(t) = 2\pi(-\beta t_0)t + (\pi\beta t_0^2 - 2\pi f_0 t_0) = 2\pi f_0' t + \phi_0'$$

Thus the resulting wave is a sine wave with constant frequency f_0' and constant phase ϕ_0'. Since amplitude is nonchanging and there is no way to measure phase, we measure the frequency and solve for t_0:

$$-\beta t_0 = f_0' \Rightarrow t_0 = -\frac{f_0'}{\beta}$$

6. In communications applications, the target bandpass signal is called an RF signal, and uses some intermediate frequency signal IF. Let $RF = 1.0$GHz, $IF = 70$MHz.

   (a) *What are the two possible choices for f_0?*
   930.0MHz, 1070.0MHz

   (b) *For each choice of f_0, what other frequencies will emerge when the 1.0GHz signal is mixed with f_0?*
   For 930.0MHz: 1930MHz; for 1070.0MHz: 2070MHz

   (c) *Find the image frequency of the lower choice of f_0.*
   For 930.0MHz: 860MHz

   (d) *Find the image frequency of the higher choice of f_0.*
   For 1070.0MHz: 1140MHz

   (e) *Does the (absolute value of the) difference between the RF and image frequencies depend on the choice of f_0 for a given f_{IR}? Does it depend on the value of f_{IF}? Can you surmise the formula for $|f_{RF} - f_{image}|$?*

      i. No; the frequencies f_0 are $f_{RF} \pm f_{IF}$ and thus the image frequencies are $f_{RF} \pm 2f_{IF}$, both lying $2f_{IF}$ away from f_{RF}, and thus independent of the choice of f_0.

      ii. Yes, as per the explanation in (i).

      iii. $|f_{RF} - f_{image}| = 2f_{IF}$, as per the explanation in (i).