(a) Find the surface area of the visible region of a sphere of radius r that an observer at height H above the surface of the sphere can see.

We'll work on the 2D-plane (and only in the first quadrant) for simplicity, w.l.o.g. due to symmetry. Assume that the sphere is centered at the origin, and that the observer is located at $(0, H + r)$. Then the sphere is bounded on the top by $y_1 = \sqrt{r^2 - x^2}$, and the visible region of the sphere is limited by some straight line, $y_2 = H + r - cx$. We know that the two lines must intersect tangentially, and this constrains the value of c. First, we take the derivatives:

$$y_1' = -\frac{x}{\sqrt{r^2 - x^2}}$$

$$y_2' = -c$$

Per the discussion above, we have the following system of equations

$$\begin{cases} y_1 = y_2 \\ y_1' = y_2' \end{cases}$$

from which we obtain that the point of intersection has a y-coordinate of

$$y_{min} = \frac{r^2}{H + r}$$

To obtain the answer, we integrate the surface area of the sphere from this minimum y value to $y_{max} = r$. The surface area of revolution (around the y-axis on the xy-plane) is

$$SA = \int 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Making the appropriate substitutions and simplifying, we obtain

$$SA = 2\pi \int_{\frac{r^2}{r+H}}^{r} \sqrt{r^2 - y^2} \sqrt{1 + \frac{y^2}{r^2 - y^2}} dy = 2\pi r \int_{\frac{r^2}{r+H}}^{r} dy = \frac{2\pi r^2 H}{r + H}$$