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1 Purpose

Geometric optics provides simple relationships between objects and their images through a spherical or parabolic lens. In particular, the thin lens equation relates focal length, image distance, and object distance through a simple inverse-sum law. This lab explores the use of symmetry of the relationship between the image and object distance to focus an image with the source and screen a given distance apart at two distinct lens positions, and how this may be used with Bessel's method (a reformulation of the thin lens equation) to experimentally determine focal length of a lens.

Secondly, two lenses are placed apart at a distance nearly the sum of their focal lengths to explore the application of lenses in a refracting telescope, and the empirical angular magnification compared to the calculated value.

2 Data

Each lens had a two-character identifier scratched onto it. From now on, these will the lenses will be consistently referred to as lenses 1 and 2, respectively.

Table 1: Lens identifiers

Lens 1 identifier	Lens 2 identifier
F1	77

2.1 Variable reference

p_{so} (corrected) source position

p_{l1} (corrected) lens position 1

p_{l2} (corrected) lens position 2

p_{sc} (corrected) screen position

f calculated trial focal length

δf uncertainty for calculated focal length

$\bar{f}$ mean calculated focal length for lens

$\delta \bar{f}$ uncertainty for mean calculated focal length for lens

l_1 length of longer tape

l_2 length of shorter tape

m_θ empirical (angular) magnification

2.2 Part A Data

Table 2: Estimated lens focal lengths

Lens 1 estimated f (cm)	Lens 2 estimated f (cm)
5.50	28.00

Table 3: Positions of source, lens, and image for lens 1

Trial	p_{so} (cm)	p_{l1} (cm)	p_{l2} (cm)	p_{sc} (cm)	f (cm)	δf (cm)
1	77.85	84.58	113.71	120.04	5.519	0.0606
2	66.98	73.69	113.91	120.04	5.643	0.0657
3	96.61	105.80	111.62	120.04	5.496	0.0311
4	87.88	96.44	113.29	120.04	5.833	0.0487
5	73.52	80.29	113.82	120.04	5.588	0.0629
6	62.31	69.05	113.96	120.04	5.698	0.0673

Source distance offset (cm)	1.80
Lens distance offset (cm)	0.00
Screen distance offset (cm)	0.00
Instrumental error (cm)	0.05
$\bar{f}$ (cm)	5.6
$\delta\bar{f}$ (cm)	0.14

Table 4: Positions of source, lens, and image for lens 2

Trial	p_{so} (cm)	p_{l1} (cm)	p_{l2} (cm)	p_{sc} (cm)	f (cm)	δf (cm)
1	1.85	40.01	83.22	120.04	25.598	0.0383
2	10.12	51.92	79.62	120.04	25.735	0.0314
3	14.50	59.95	76.79	120.04	25.713	0.0261
4	8.29	49.00	79.98	120.04	25.790	0.0329
5	3.81	42.18	82.16	120.04	25.619	0.0370
6	17.88	64.70	72.10	120.04	25.406	0.0214

Source distance offset (cm)	1.80
Lens distance offset (cm)	0.00
Screen distance offset (cm)	0.00
Instrumental error (cm)	0.05
$\bar{f}$ (cm)	25.64
$\delta\bar{f}$ (cm)	0.08

2.3 Part B Data

Table 5: Expected angular magnification

Expected angular magnification (cm)	-4.6
Error for mean angular magnification (cm)	0.1

Table 6: Refracting telescope tape lengths

Trial	Distance between lenses (cm)	l_1 (cm)	l_2 (cm)	Trial m_θ
1	31.00	82.20	20.90	3.933
2	31.00	92.10	18.90	4.873
3	31.10	58.50	11.55	5.065

Mean angular magnification (cm)	-4.62
Error for mean angular magnification (cm)	0.04

3 Calculations

Note that the full precision of each measurement is kept until the final result of each calculation, in which the results are rounded to the proper number of significant digits.

3.1 Instrumental Error

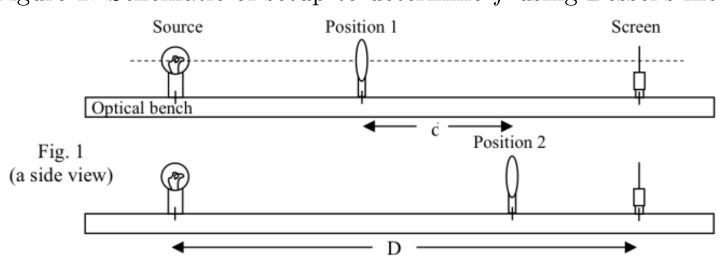
Measurements for (Part A) of this lab were taken with the linear scale on the optical bench, and measurements for (Part B) of this experiment (lengths of tape) were taken using the meter stick. Both instruments had markings to the nearest 0.1cm, so the instrumental uncertainty is $\delta S = \pm 0.05$cm for every reading. Distance measurements (e.g., l_1, l_2) are the function of two uncertainties and the error $\delta l_1 = \delta l_2 = 0.05\sqrt{2}$cm reflects this.

3.2 Sample mean and random error

For each position measurement p_x (i.e., p_{sc}, p_{so}, p_{l1}, p_{l2}), the mean $\bar{p}_x$ and standard error σ_{p_x} may not be calculated, as these measurement are not centered around a mean value; the only mean and standard deviation may be calculated for the focal lengths, as these should be centered around the true value of the focal length of the lens.

3.3 Calculation of the focal length using Bessel's method

Figure 1: Schematic of setup to determine f using Bessel's method



The focal length of the lenses is calculated using Bessel's method. Given the setup indicated in (Figure), Bessel's method uses (Equation 1) to solve for the focal length f of the lens, and is derived in (Section 6.1).

$$f = \frac{D^2 - d^2}{4D} \tag{1}$$

In short, the source and screen are set up more than four focal lengths away from each other on the optical bench, and their positions (p_{so} and p_{sc}, respectively) measured. There should be two positions that the lens may be placed (p_{l1} and

p_{l2}), such that the image of the source is focused on the screen. In this setup, we make the substitutions

$$D = |p_{sc} - p_{so}|$$

$$d = |p_{l2} - p_{l1}|$$

to obtain

$$f = \frac{|p_{sc} - p_{so}|^2 - |p_{l2} - p_{l1}|^2}{4|p_{sc} - p_{so}|}$$

Since the measurements for each trial are not centered around any particular value and may vary dramatically, this method may not be used to calculate average $\bar{f}$ using average values for each position measurement, but rather only the focal length based on the measurements for a single trial. The average calculated focal length may be found as a mean of the sample values:

$$\bar{f} = \frac{1}{n}\sum_{i=1}^{n} f_i$$

3.3.1 Error propagation

The measurements p_{sc} and p_{so} are independent, as the distance between source and image is arbitrarily chosen. However, the values for p_{l1}, p_{l2}, and D are dependent, as the arbitrary choice of D determines the two possible lens positions. This inspires a two-part error propagation calculation.

$$\delta D = \sqrt{\left(\frac{\partial D}{\partial p_{so}}\delta p_{so}\right)^2 + \left(\frac{\partial D}{\partial p_{sc}}\delta p_{sc}\right)^2}$$

$$\delta f = \left|\frac{\partial f}{\partial D}\delta D\right| + \left|\frac{\partial f}{\partial p_{l1}}\delta p_{l1}\right| + \left|\frac{\partial f}{\partial p_{l2}}\delta p_{l2}\right|$$

In this case, the partial derivatives are (signs are not important because the error is always positive):

$$\frac{\partial D}{\partial p_{so}} = \frac{\partial D}{\partial p_{sc}} = 1$$

$$\frac{\partial f}{\partial D} = \frac{D^2 + d^2}{4D^2}$$

$$\frac{\partial f}{\partial p_{l1}} = \frac{\partial f}{\partial p_{l2}} = \frac{d}{2D}$$

Making all substitutions, the error calculation for f is

$$\delta f = \left|\frac{D^2 + d^2}{4D^2}\sqrt{\delta p_{so}^2 + \delta p_{sc}^2}\right| + \left|\frac{d}{2D}\delta p_{l1}\right| + \left|\frac{d}{2D}\delta p_{l2}\right|$$

This is the error calculation for a single f_i calculation, where δp_x is the instrumental error for the p_x measurement. Note that since all positions were measured with the same instrument, $\delta p_x = \delta p$ are equal, that all terms inside the absolute value bars are positive, and the first term includes the f calculation, and thus the calculation may be simplified to

$$\delta f = \left(\left(f + \frac{d^2}{2D} \right) \sqrt{2} + d \right) \frac{\delta p}{D}$$

Since each sample focal length calculation f_i is independently calculated, the error for the mean $\delta \bar{f}$ is the RMS of the f_i errors, i.e.,

$$\delta \bar{f} = \sqrt{\sum_{i=1}^{n} (\delta f_i)^2}$$

3.3.2 Sample Bessel's law focal length and error propagation calculations

The calculation for the focal length for sample 1 using Bessel's law is shown below.

$$f_1 = \frac{(120.04\text{cm} - 77.85\text{cm})^2 + (113.71\text{cm} - 84.58\text{cm})^2}{4(120.04\text{cm} - 77.85\text{cm})} = 5.\bar{5}19\text{cm}$$

The calculation for the error for the focal length for sample 1 is shown below.

$$\delta f_1 = \left(\left(5.519\text{cm} + \frac{(113.71cm - 84.58\text{cm})^2}{4(120.04cm - 77.85\text{cm})} \right) \sqrt{2} + (113.71\text{cm} - 84.58\text{cm}) \right)$$
$$\times \frac{0.05\text{cm}}{120.04\text{cm} - 77.85\text{cm}} = 0.06\text{cm}$$

The average and average uncertainty are as calculated below.

$$\bar{f} = \frac{1}{6}(5.\bar{5}19\text{cm} + 5.\bar{6}43\text{cm} + 5.\bar{4}96\text{cm} + 5.\bar{8}33\text{cm} + 5.\bar{5}88\text{cm} + 5.\bar{6}98\text{cm}) = 5.\bar{6}30\text{cm}$$

$$\delta \bar{f} = \sqrt{(0.0\bar{6}06\text{cm})^2 + (0.0\bar{6}57\text{cm})^2 + \cdots (0.0\bar{6}73\text{cm})^2} = 0.14\text{cm}$$

3.4 Calculation of angular magnification using focal lengths

The expected angular magnification for the telescope is

$$m_\theta = -\frac{f_{obj}}{f_{eye}}$$

3.4.1 Error propagation

The focal lengths of the two lenses are independent of one another, so

$$\delta m_\theta = \sqrt{\left(\frac{\partial m_\theta}{\partial f_{obj}} \delta f_{obj} \right)^2 + \left(\frac{\delta m_\theta}{\delta f_{eye}} \delta f_{eye} \right)^2} = \sqrt{\left(\frac{1}{f_{eye}} \delta f_{obj} \right)^2 + \left(\frac{f_{obj}}{f_{eye}^2} \delta f_{eye} \right)^2}$$

where f_{obj}, f_{eye} are obtained in (Section 3.3.1).

3.4.2 Calculation for angular magnification and error propagation

$$m_\theta = -\frac{5.\bar{6}30\text{cm}}{25.6\bar{4}4\text{cm}} = -4.\bar{5}55$$

$$\delta m_\theta = \sqrt{\left(\frac{1}{5.630cm}0.0777cm\right)^2 + \left(\frac{25.644cm}{(5.630cm)^2}0.141cm\right)^2} = 0.11$$

3.5 Calculation of angular magnification using empirical tape lengths

The angular magnification for the telescope is the ratio of the lengths of tape (such that the shorter piece of tape viewed through the telescope appears as long as the longer piece of tape viewed without the telescope). Denote this $m_{\theta\text{emp}}$.

$$m_{\theta\text{emp}} = \frac{l_1}{l_2}$$

A mean is taken over the three trials for the empirical angular magnifications to get the mean empirical angular magnifications.

A percent error calculation is performed to check the closeness of this angular magnification from the angular magnification calculated using focal lengths.

$$\%\text{ Err.} = \frac{|\bar{m}_{\theta\text{emp}} - m_\theta|}{m_\theta} \times 100\%$$

3.5.1 Error propagation for calculation of angular magnification using empirical tape lengths

Since this equation is of the same form as the previous calculation for angular magnification, the error propagation (for a single trial) is of the same form.

$$\delta m_{\theta\text{emp}} = \sqrt{\left(\frac{1}{l_2}\delta l_1\right)^2 + \left(\frac{l_1}{l_2^2}\delta l_2\right)^2}$$

Again, the error for the mean value is the RMS of the trial errors.

3.5.2 Sample calculation of empirical angular magnification and error propagation

For trial 1, the empirical angular magnification calculation is

$$m_{\theta\text{emp}_1} = \frac{82.20\text{cm}}{20.90\text{cm}} = 3.9\bar{3}3$$

Note that $\delta l_1 = \delta l_2 = 0.05\sqrt{2}$cm, since it is a distance measurement and not a single reading. The error for this trial is thus

$$\delta m_{\theta\text{emp}_1} = \sqrt{\left(\frac{1}{20.9\text{cm}}0.07\text{cm}\right)^2 + \left(\frac{82.2\text{cm}}{(20.9\text{cm})^2}0.07\text{cm}\right)^2} = 0.014$$

The mean empirical angular magnification is

$$\bar{m}_{\theta \mathrm{emp}} = \frac{1}{3}(3.933 + 4.873 + 5.065) = 4.6\bar{2}4$$

The error for the mean empirical angular magnification is

$$\delta \bar{m}_{\theta \mathrm{emp}} = \sqrt{0.0137^2 + 0.0186^2 + 0.0316^2} = 0.04$$

The percent error calculation is:

$$\% \text{ error} = \frac{|4.6\bar{2}4 - 4.\bar{5}55|}{4.\bar{5}55} = 1.5\%$$

4 Results summary

The results from Part A of the lab are summarized in (Table).

Table 7: Focal lengths summary

Lens	f (cm)
1	$5.6 \pm 0.14 cm$
2	$25.64 \pm 0.08 cm$

The results from Part B are summarized in (Table).

Table 8: Angular magnification summary

Method	Angular magnification
Using focal lengths from Part A	-4.6 ± 0.1
Using tape lengths from Part B	-4.62 ± 0.04

% error	1.5%

5 Error analysis

This section should be precluded by the fact that all measurements were performed at a certain configuration of the lens(es) with objects that are visually determined to be in focus or of equal length. There is inherent error in this, as the human eye is not especially precise with determining either of these.

5.1 Part A sources of error

In the original estimate of focal lengths, an estimation takes place assuming that the focal length is approximately the length of a lens from a screen that it aims to focus an image onto. The derivation for this estimation is present in Section 7.2, as well as an argument for why it is an overestimation – thus there is an inherent modeling error. Moreover, there is also large error introduced by the method used to measure this distance; e.g., if the meter stick is not held perfectly perpendicular to the ground, if the image is not precisely focused, or if the reading is not taken at the center of the lens (as was hard to do, as one of the lenses has a short focal length and thus was positioned very near the ground), then these are all sources of random error. The reasons why procedural precautions were not made is because this was only a rough estimate for the next stages. Thus no proper error analysis is made for this section.

There is an offset for each of the lens, screen, and source. To attempt to measure this offset, we held a straight edge (meter stick) flat on the surfaces of the source and screen, perpendicular to the optical bench, and attempted to read the difference between the surface's position and the optical bench's reading. For the lens and the screen (which is almost flat; a sheet of paper), we were unable to measure a discernible difference between the two values, and made the (reasonable) assumption that the error contributed by these offsets is very small compared to the distances between them (which would ultimately be used in all of the calculations). However, the source offset was significant, 1.80cm, and was added to each source measurement. These small offsets may add some systematic error to the calculations, but we performed the a correction to the best of our knowledge with only the source offsets. A better method to measure offsets would have been to use a more precise measuring tool, such as calipers.

There was a significant amount of light pollution from other groups' flashlights, which made it more difficult to determine precisely at what lens position the image was clearly focused, which may introduce some degree of random error.

5.2 Part B sources of error

The procedure failed to provide an exact measure for the distance between the two lenses (only dictating that the two lenses be slightly less than the sum of the two focal lengths apart). We experimented with lenses 31.0cm to 31.1cm away, with a large standard deviation of results. It is difficult to say whether this small difference in lens distance may have contributed a significant error, or if it was the inconsistency of the human eye to determine when the two pieces of tape were indeed the same length, one being viewed inside the telescope and one viewed outside. The latter is much more likely to be the overwhelmingly larger source of error, as there is no way to put both images side-to-side within the same viewing frame, which is how it is easiest to compare objects visually; instead, either the telescope has to be moved into view and overwriting the non-telescope image, or both eyes may be used simultaneously, but it is difficult to compare items when each eye is seeing a different image.

Another possible source of random error is the changing of viewing distance from the telescope wielder to the pieces of tape. This was not measured during the experiment, but there was a wide range of viewing distances used (in an attempt to make the visual comparison between the lengths easier), and its effect is not measured or known. Theoretically, this should make no difference to the outcome if the eye was accurate in its comparison, but it may likely have introduced differences in perception that lead to random error.

6 Conclusion

By using Bessel's method, the focal lengths of the eyepiece lens and the object lens were 5.6 ± 0.14cm and 25.64 ± 0.08cm. Although the true focal lengths of these lenses is unknown, the small standard deviations (on the order of 1mm) indicate that Bessel's method produces fairly reliable results. The initial estimated focal length for lens 1 (5.50cm) is captured within the error bars, but the estimated focal length for lens 2 (28.00cm) is not, but off by less than 10%. Given that the initial estimate is really a rough estimate, this error is acceptable.

This result is further strengthened by Part B's verification of the angular magnification of a telescope formed using the lenses. The expected angular magnification calculated using the focal lengths in Part A is -4.6 ± 0.1, while the average angular magnification calculated using tape lengths is -4.62 ± 0.08. yielding only a 1.5% error. While these error margins do not overlap, the magnification factors are very close (the difference is only 0.2) and the errors are very small, so this may be interpreted as both methods converging towards a similar, precise result.

There is, as expected for a method with large visual estimation and uncertainty, a large standard deviation between trials, and thus many trials are recommended to obtain a more precise result through this method.

7 Answers to questions

7.1 Derivation of Bessel's law

If the distance between the source and object is fixed, then the distance p between the lens and the object, and the distance i between the lens and the projected image is related by the thin lens equation

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{i} \tag{2}$$

Since the relationship between p and i is symmetric, if the lens focuses the image when it is at distance p from the source, it will also focus it at length i from the source. We have two other relations that can be determined from (Figure): firstly, since the images are real and the distance between the source and image is the sum of p and i,

$$p + i = D$$

Since the lens may focus an image at position p or position i from the source, the distance between the two focusing positions of the lenses is

$$p - i = d$$

Solving for f from (Equation 2), we get

$$f = \frac{1}{\frac{1}{p} + \frac{1}{i}} = \frac{pi}{p + i}$$

With some algebraic manipulation:

$$pi = \frac{(p^2 + 2pi + i^2) - (p^2 - 2pi + i^2)}{4} = \frac{(p + i)^2 - (p - i)^2}{4} = \frac{D^2 - d^2}{4}$$

Substituting in for pi, we get Bessel's equation for focal length

$$f = \frac{D^2 - d^2}{4D}$$

We can rearrange Bessel's equation into the following quadratic inequality, given that $D, d > 0$ ($D > 0$ clearly, and $d > 0$ for the separation of lens positions). This inequality demonstrates the reason for requiring that D is greater than four times the focal length.

$$D^2 - 4fD - d^2 = 0 \Rightarrow D^2 - 4fD > 0 \Rightarrow D(D - 4f) > 0 \Rightarrow D > 4f$$

(Bessel's equation also works in the degenerate case of $d = 0$, corresponding to the scenario in which $i = p$, in which $f = \frac{D^2 - 0}{4D} = \frac{D}{4}$, as expected; however, the experimental procedure for this lab expected two lens positions producing clear images to use the nontrivial form of Bessel's equation.)

7.2 Initial overestimation of focal length

To estimate focal length, a sharp image of an object far away was produced – for this experiment, an image of the overhead lights was produced on the floor. We used the approximation that the object was at infinite distance (i.e., $p = \infty$) to make the following approximation from the thin lens equation:

$$\frac{1}{f} = \lim_{p \to \infty} \left(\frac{1}{p} \right) + \frac{1}{i} \Rightarrow \frac{1}{f} = 0 + \frac{1}{i} \Rightarrow f \approx i$$

Of course, p is finite, especially considering the measurable distance to the ceiling lights, so $\frac{1}{p} = \epsilon > 0$. Thus, a more accurate representation using the thin lens equation is

$$\frac{1}{f} = \frac{1}{i} + \epsilon \Rightarrow f = \frac{1}{\frac{1}{i} + \epsilon} < i$$

Therefore i is an overestimation for the focal length. Since the ratio $\frac{p}{i}$ was very large and this approximation was only used to get a rough estimate of the focal length to be more accurately measured in the next part of the procedure, this approximation should not contribute any error to the final results of this lab.