1. Purpose

In Part A of the procedure, the diameter and thickness of a cylindrical disk were measured using a metric scale and a vernier caliper. Since the measuring tools have differing instrumental uncertainties, this allows the comparison of the effect of instrumental uncertainty on the volume calculated from these measurements.

In Part B of the procedure, the dimensions (length and width) and mass of a rectangular plate and a cylindrical disk were measured, and their densities were calculated and compared to each other. A micrometer was used for thickness measurements, a vernier caliper was used for other linear measurements, and a digital balance used for mass measurements. This is compared to the literature value of the material to determine the method's accuracy.

Part C involves measuring the lengths of two moderately-sized ($n = 20$) samples of small acrylic pieces. The report uses a hypothesis test to determine whether the two samples come from the same sample within a 95% confidence interval.

2. Data

2.1. Part A data

Table 1: Diameter of the cylindrical disk (metric ruler)

Right Reading (cm)	Left Reading (cm)	Length (cm)
4.84	1.00	3.84
6.84	3.00	3.84
7.91	4.10	3.81
12.95	9.15	3.80
9.38	5.45	3.93
9.95	6.13	3.82

Instrumental Error (cm)	0.04
Random Error (cm)	0.02
Diameter (cm)	3.84 ± 0.04

Table 2: Thickness of the cylindrical disk (metric ruler)

Right Reading (cm)	Left Reading (cm)	Length (cm)
6.00	5.70	0.31
5.61	5.30	0.31
11.10	10.80	0.30
8.61	8.29	0.32
7.60	7.30	0.30
7.20	6.90	0.30

Instrumental Error (cm)	0.04
Random Error (cm)	0.003
Diameter (cm)	0.30 ± 0.04

Table 3. Diameter of the cylindrical disk (vernier caliper)

Length (cm)	Corrected Length (cm)
3.850	3.850
3.844	3.844
3.854	3.854
3.854	3.854
3.864	3.864
3.852	3.852

Zero Error (cm)	0.000
Instrumental Error (cm)	0.002
Random Error (cm)	0.003
Diameter (cm)	3.853 ± 0.003

Table 4. Thickness of the cylindrical disk (vernier caliper)

Length (cm)	Corrected Length (cm)
0.324	0.324
0.320	0.320
0.324	0.324
0.330	0.330
0.324	0.324
0.327	0.327

Zero Error (cm)	0.000
Instrumental Error (cm)	0.002
Random Error (cm)	0.001
Diameter (cm)	0.325 ± 0.001

2.2. Part B data

Table 5. Length of the rectangular object (vernier caliper)

Length (cm)	Corrected Length (cm)
3.824	3.824
3.826	3.826
3.826	3.826
3.842	3.842
3.844	3.844
3.826	3.826

Zero Error (cm)	0.000
Instrumental Error (cm)	0.002
Random Error (cm)	0.004
Diameter (cm)	3.831 ± 0.004

Table 6. Width of the rectangular object (vernier caliper)

Length (cm)	Corrected Length (cm)
5.060	5.060
5.058	5.058
5.050	5.050
5.040	5.040
5.050	5.050
5.042	5.042

Zero Error (cm)	0.000
Instrumental Error (cm)	0.002
Random Error (cm)	0.003
Diameter (cm)	5.050 ± 0.003

Table 7. Thickness of the rectangular object (micrometer)

Length (cm)	Corrected Length (cm)
0.3111	0.3111
0.3165	0.3165
0.3151	0.3151
0.3092	0.3092
0.3154	0.3154
0.3156	0.3156

Zero Error (cm)	0.0000
Instrumental Error (cm)	0.0005
Random Error (cm)	0.0012
Diameter (cm)	0.314 ± 0.0012

Table 8. Thickness of the cylindrical disk (micrometer)

Length (cm)	Corrected Length (cm)
0.3239	0.3239
0.3172	0.3172
0.3302	0.3302
0.3179	0.3179
0.3251	0.3251
0.3185	0.3185

Zero Error (cm)	0.0000
Instrumental Error (cm)	0.0005
Random Error (cm)	0.002
Diameter (cm)	0.322 ± 0.002

Table 9. Mass of the cylindrical disk (digital balance)

Length (cm)	Corrected Length (cm)
9.9	9.9
9.9	9.9
9.9	9.9
9.9	9.9
9.9	9.9
9.9	9.9

Zero Error (cm)	0.00
Instrumental Error (cm)	0.05
Random Error (cm)	0.00
Diameter (cm)	9.9 ± 0.05

Table 10. Mass of the rectangular object (digital balance)

Length (cm)	Corrected Length (cm)
16.1	16.1
16.1	16.1
16.1	16.1
16.1	16.1
16.1	16.1
16.1	16.1

Zero Error (cm)	0.00
Instrumental Error (cm)	0.05
Random Error (cm)	0.00
Diameter (cm)	16.1 ± 0.05

2.3. Part C data

Table 11. Length of acrylic pieces in Bottle #4 (micrometer)

Length (cm)	Corrected Length (cm)
1.3875	1.3875
1.3972	1.3972
1.4040	1.4040
1.2975	1.2975
1.3560	1.3560
1.2930	1.2930
1.3415	1.3415
1.3980	1.3980
1.3440	1.3440
1.2875	1.2875
1.3450	1.3450
1.3962	1.3962
1.3445	1.3445
1.2914	1.2914
1.4395	1.4395
1.2945	1.2945
1.3920	1.3920
1.3421	1.3421
1.2940	1.2940
1.3130	1.3130

Zero Error (cm)	0.0000
Instrumental Error (cm)	0.0005
Bottle 4 STDEV (cm)	0.046943
Bottle 4 STDOM (cm)	0.019164
Bottle 4 length (cm)	1.35 ± 0.05

Table 12. Length of acrylic pieces in Bottle #5 (micrometer)

Length (cm)	Corrected Length (cm)
1.3458	1.3458
1.3500	1.3500
1.4320	1.4320
1.3391	1.3391
1.3597	1.3597
1.3575	1.3575
1.3971	1.3971
1.3380	1.3380
1.4410	1.4410
1.2318	1.2318
1.3750	1.3750
1.3470	1.3470
1.2718	1.2718
1.3371	1.3371
1.3252	1.3252
1.3746	1.3746
1.3641	1.3641
1.3472	1.3472
1.3781	1.3781
1.2295	1.2295

Zero Error (cm)	0.0000
Instrumental Error (cm)	0.0005
Bottle 5 STDEV (cm)	0.053855
Bottle 5 STDOM (cm)	0.021986
Bottle 5 length (cm)	1.35 ± 0.05

3. Calculation

3.1. Instrument errors

The metric ruler has markings to the nearest 0.05cm and involves visual estimation, so the instrumental error for a single reading is $\delta S_{read} = \pm 0.025\mathrm{cm}$. A length measurement using the metric ruler is a function of two (independent) readings. Thus the error for a single length measurement δS is calculated from the errors of the left and right instrumental readings of the metric ruler:

$$\delta S = \sqrt{\delta S_{left}^2 + \delta S_{right}^2} = \sqrt{(0.025\mathrm{cm})^2 + (0.025\mathrm{cm})^2} = 0.0354\mathrm{cm} = 0.04\mathrm{cm}$$

The vernier caliper has markings to the nearest 0.002cm, and reading it involves choosing the closest matching margin. Since there is no approximation between the markings, the instrumental uncertainty is $\delta S = \pm 0.002\mathrm{cm}$.

The micrometer has markings to the nearest 0.001cm. Reading it involves visual estimation between markings, so the instrumental error for a reading is $\delta S = \pm 0.0005\mathrm{cm}$.

The digital balance reports data to 0.1g. This means that the instrumental error is $\delta S = \pm 0.05g$.

None of the instruments had a measurable zero error (i.e., the zero errors were all 0 to all significant figures), so no offset corrections were performed on measurements.

3.2. Sample mean and random error

For each sample of measurements, the measurement is recorded with both a “best value” and a specified random error. The “best value” is the mean $\bar{x}$, and the random error is reported as standard deviation of the mean (STDOM) $\sigma_{\bar{x}}$. These are discussed in (3.5. Part C calculations) with sample calculations.

3.3. Part A calculations

The volume of the cylindrical disk is calculated with the volume formula:

$$V_{disk} = \pi \left(\frac{d}{2}\right)^2 h$$

where d is the disk diameter and h is the cylinder height. The error propagation formula is:

$$\delta V = \sqrt{\left(\frac{\partial V}{\partial d}\delta d\right)^2 + \left(\frac{\partial V}{\partial h}\delta h\right)^2} = \sqrt{\left(\frac{\pi d h}{2}\delta d\right)^2 + \left(\frac{\pi d^2}{4}\delta h\right)^2}$$

where δd and δh are the larger of the instrumental and random error for diameter and height measurements, respectively. The 2-norm is used because the measurements are independent.

Sample calculations

For measurements using the metric ruler:

$$V_{disk} = \pi\left(\frac{3.84\text{cm}}{2}\right)^2 \cdot 0.30\text{cm} = 3.5\text{cm}^3$$
$$\delta V = \sqrt{\left(\frac{\pi \cdot 3.84\text{cm} \cdot 0.30\text{cm}}{2} \cdot 0.04\text{cm}\right)^2 + \left(\frac{\pi (3.84\text{cm})^2}{4} \cdot 0.04\text{cm}\right)^2} = 0.414\text{cm}^3$$

In this calculation, the instrumental error (0.04cm) is larger than the random error and therefore used for δd and δh. For measurements using the vernier caliper:

$$V_{disk} = \pi\left(\frac{3.853\text{cm}}{2}\right)^2 \cdot 0.325\text{cm} = 3.79\text{cm}^3$$
$$\delta V = \sqrt{\left(\frac{\pi \cdot 3.853\text{cm} \cdot 0.325\text{cm}}{2} \cdot 0.003\text{cm}\right)^2 + \left(\frac{\pi (3.853\text{cm})^2}{4} \cdot 0.002\text{cm}\right)^2} = 0.0239\text{cm}^3$$

Again, the larger of the instrumental and random errors are chosen; this is the common theme for error propagation and will be done without explanation in any following error propagation problems.

Change in uncertainty

Define the relative change of uncertainty be:

$$\text{rel. change uncertainty} = \frac{\text{new uncertainty} - \text{old uncertainty}}{\text{old uncertainty}} \times 100\%$$

Then, the relative change in uncertainty by switching from the metric ruler to the vernier caliper is:

$$\text{rel change }\% = \frac{0.002\text{cm} - 0.035\text{cm}}{0.035\text{cm}} \times 100\% = -94\%$$

since the error for a ruler measurement is 0.035cm and that of the vernier caliper is 0.002cm. The relative change in the volume calculation is:

$$\text{rel change }\% = \frac{0.0239cm - 0.414cm}{0.414cm} \times 100\% = -94.2\%$$

Interestingly, this is essentially equal to the relative change in uncertainty of the measuring instrument.

3.4. Part B calculations

The density of the rectangular object is calculated using the following formula:

$$\rho = \frac{m}{lwh}$$

where ρ is the calculated density, m is the measured mass, l, w, and h are measured length, width, and height, respectively. The error propagation for the density of the rectangular solid is:

$$\delta\rho = \sqrt{\left(\frac{\partial\rho}{\partial m}\delta m\right)^2 + \left(\frac{\partial\rho}{\partial l}\delta l\right)^2 + \left(\frac{\partial\rho}{\partial w}\delta w\right)^2 + \left(\frac{\partial\rho}{\partial h}\delta h\right)^2}$$
$$= \sqrt{\left(\frac{1}{lwh}\delta m\right)^2 + \left(-\frac{m}{l^2wh}\delta l\right)^2 + \left(-\frac{m}{lw^2h}\delta w\right)^2 + \left(-\frac{m}{lwh^2}\delta h\right)^2}$$

The density of the cylindrical disk is calculated using the following formula:

$$\rho = \frac{m}{\pi\left(\frac{d}{2}\right)^2 h}$$

where ρ is the calculated density, m is the measured mass, d is the measured diameter, and h is the measured thickness. The error propagation for the density of the cylindrical disk is:

$$\delta\rho = \sqrt{\left(\frac{\partial\rho}{\partial m}\delta m\right)^2 + \left(\frac{\partial\rho}{\partial d}\delta d\right)^2 + \left(\frac{\partial\rho}{\partial h}\delta h\right)^2}$$
$$= \sqrt{\left(\frac{4}{\pi d^2 h}\delta m\right)^2 + \left(-\frac{8m}{\pi d^3 h}\delta d\right)^2 + \left(-\frac{4m}{\pi d^2 h^2}\delta h\right)^2}$$

since the measurements for mass, diameter, and thickness are all independent.

Sample calculations

For the density of the rectangular object:

$$\rho = \frac{16.1\text{g}}{3.831\text{cm}\cdot 5.050\text{cm}\cdot 0.314\text{cm}} = 2.65\frac{\text{g}}{\text{cm}^3}$$

$$\delta\rho = \sqrt{\left(\frac{1}{3.831\text{cm}\cdot 5.050\text{cm}\cdot 0.314\text{cm}}\cdot 0.05g\right)^2 + \cdots + \left(\frac{16.1\text{g}}{3.831\text{cm}\cdot 5.050\text{cm}\cdot (0.314\text{cm})^2}\cdot 0.001\text{cm}\right)^2} = 0.07\frac{\text{g}}{\text{cm}^3}$$

For the density of the cylindrical disk:

$$\rho = \frac{9.9\text{g}}{\pi\left(\frac{3.853\text{cm}}{2}\right)^2 0.314\text{cm}} = 2.6\frac{\text{g}}{\text{cm}^3}$$

$$\delta\rho = \sqrt{\left(\frac{4}{\pi(3.853\text{cm})^2\cdot 0.3185\text{cm}}\cdot 0.05\text{g}\right)^2 + \cdots + \left(\frac{4\cdot 9.9\text{g}}{\pi(3.853\text{cm})^2\cdot (0.3185\text{cm})^2}\cdot 0.002\text{cm}\right)^2} = 0.03\frac{\text{g}}{\text{cm}^3}$$

Comparison with the literature value

The aluminum alloy composition of the cylindrical disk and rectangular object is known to be Alloy 6061, which has a density of 2.70g[1]. Both densities are close to (within $0.10\frac{\mathrm{g}}{\mathrm{cm}^3}$) but less than this literature value. For the rectangular object, it falls within one standard deviation of the mean; but for the cylindrical disk, it is roughly three standard deviations above the mean.

Because of their closeness to the literature value, the densities of the rectangular object and the cylindrical disk have a small percent error when compared to the disk.

$$\%\text{ error} = \frac{|\text{empirical}-\text{literature}|}{\text{literature}} \times 100\%$$
$$\%\text{ error}_{\text{rect}} = \frac{|2.65\frac{\mathrm{g}}{\mathrm{cm}^3} - 2.70\frac{\mathrm{g}}{\mathrm{cm}^3}|}{2.70\frac{\mathrm{g}}{\mathrm{cm}^3}} \times 100\% = 1.9\%$$
$$\%\text{ error}_{\text{cylinder}} = \frac{|2.6\frac{\mathrm{g}}{\mathrm{cm}^3} - 2.70\frac{\mathrm{g}}{\mathrm{cm}^3}|}{2.70\frac{\mathrm{g}}{\mathrm{cm}^3}} \times 100\% = 4\%$$

A graphic of the comparison of the literature and calculated values is shown in (Results, Figure 1).

3.5. Part C calculations

For a sample x, the mean $\bar{x}$, the sample variance S_x^2, the standard deviation S_x, and the standard deviation of the mean (STDOM) $\sigma_{\bar{x}}$ are given by the following formulas:

$$\bar{x} = \frac{1}{N_x}\sum_{i=1}^{N_x} x_i$$
$$S_x^2 = \frac{1}{N_x - 1}\sum_{i=1}^{N_x}(x_i - \bar{x})^2$$
$$S_x = \sqrt{S_x^2}$$
$$\sigma_{\bar{x}} = \frac{S_x}{\sqrt{N_x}}$$

Sample calculations

Sample calculations for sample bottle #4 (sample x) are shown below.

$$\bar{x} = \frac{1.3875\mathrm{cm} + 1.3972\mathrm{cm} + \cdots 1.3130\mathrm{cm}}{20} = 1.3479\mathrm{cm}$$
$$S_x^2 = \frac{(1.3875\mathrm{cm} - 1.3479\mathrm{cm})^2 + (1.3972\mathrm{cm} - 1.3479\mathrm{cm})^2 + \cdots (1.3130\mathrm{cm} - 1.3479\mathrm{cm})^2}{19} = 0.0022036\mathrm{cm}^2$$
$$S_x = \sqrt{0.022036\mathrm{cm}^2} = 0.046943\mathrm{cm}$$

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[1] ASM Handbook, Volume 2: Properties and Selection: Nonferrous Alloys and Special-Purpose Materials ASM Handbook Committee, p 102 DOI: 10.1361/asmhba0001060

$$\sigma_{\bar{x}} = \frac{0.046943\text{cm}}{\sqrt{20}} = 0.010497\text{cm}$$

For sample bottle #5 (sample y), the respective values are (calculations not shown here):

$$\bar{y} = 1.3471\text{cm}$$
$$S_y^2 = 0.0029003\text{cm}^2$$
$$S_y = 0.053855\text{cm}$$
$$\sigma_{\bar{y}} = 0.012042\text{cm}$$

Two-sample t-test

See (Questions, 3) for a brief discussion on the shape of the sample distributions. The two-sample t-statistic is calculated as follows for two samples:

$$t = \frac{|\bar{x}-\bar{y}|}{\sqrt{\sigma_{\bar{x}}^2+\sigma_{\bar{y}}^2}}$$

For the two samples, #4 (x) and #5 (y), the t-statistic is calculated this way:

$$t = \frac{|1.3479\text{cm}-1.3471\text{cm}|}{\sqrt{(0.010497\text{cm})^2+(0.012042\text{cm})^2}} = 0.05009$$

Since $t << 1.96$, there is a strong suggestion the means of these two distributions is the same. See (Questions, 4) for a more complete discussion.

4. Results

Table 13. Calculated volumes and errors using metric ruler and vernier caliper

Measuring instrument	Instrumental Error (cm)	Reported Volume (cm^3)
metric ruler	0.04	3.5 ± 0.4
vernier caliper	0.002	3.79 ± 0.02

Table 14. Calculated densities and errors of the rectangular and cylindrical objects

Object	Density ($\frac{g}{cm^3}$)
cylindrical disk	2.65 0.07
rectangular object	2.7 0.03

Literature density ($\frac{g}{cm^3}$)	2.70

Figure 1 demonstrates that the calculated densities for the two samples are very close to the literature value. It also shows how the range for random error for the rectangular object captures the literature value, while the smaller random error for the cylindrical disk (which also has a lower mean) does not.

Figure 1. Measured densities relative to base density

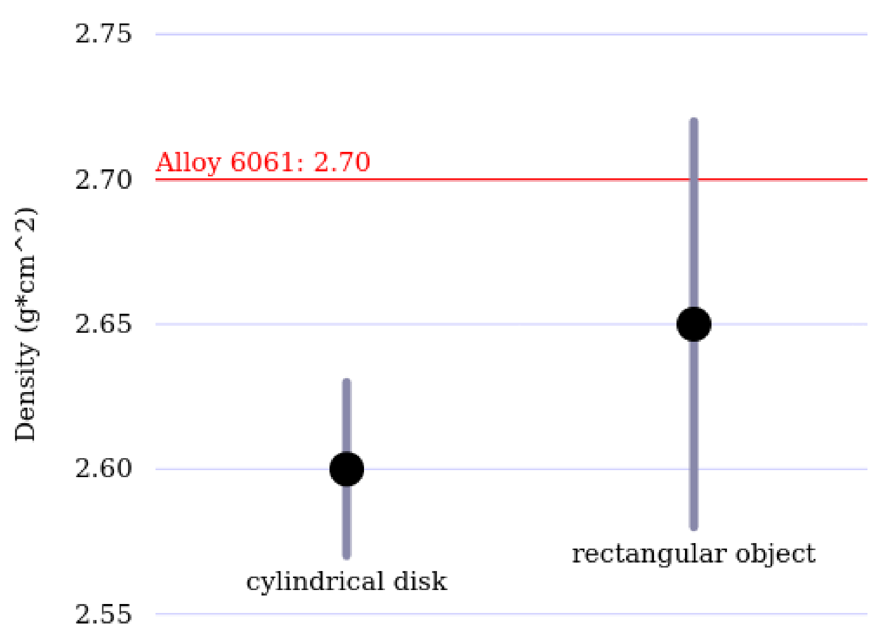


Table 15. Descriptive statistics of the samples of acrylic pieces

Container	Sample Mean (cm)	Sample Variance (cm)	Standard Deviation (cm)	Standard Deviation of the Mean (cm)
4	1.3479	0.0022036	0.0469428	0.010497
5	1.3471	0.00290038	0.0538552	0.012042

The calculated t-statistic for the two-sample t-test is 0.0528. Since $t < 1.96$, we fail to reject the null hypothesis at a 95% confidence interval.

The following two figures demonstrate. Their shapes are discussed in (Questions, 3), and their analysis is provided in (Questions, 4). It is apparent that the distributions are not perfectly normal (i.e., multiple peaks, some gaps in the distribution), but they are both not heavily skewed and the peak(s) are concentrated in the centers of the distribution. Also, the distributions have similar center, spread, and counts per bin. This suggests that the samples come from a similar parent population.

Figure 2. Histogram of length of acrylic pieces in bottle #4 (cm)

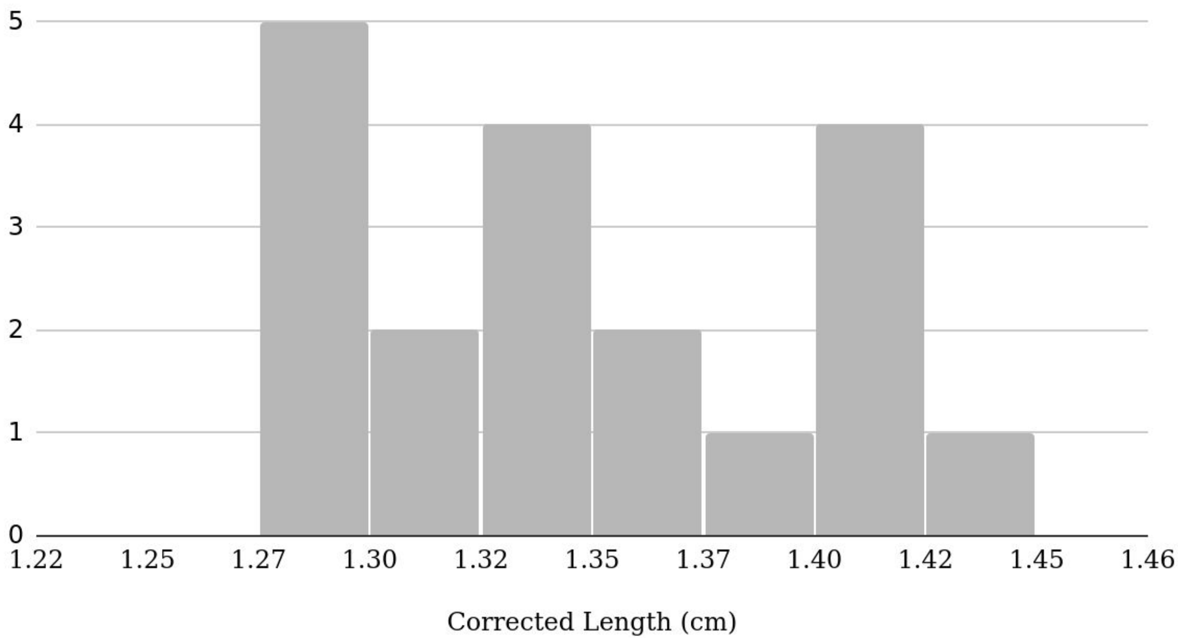
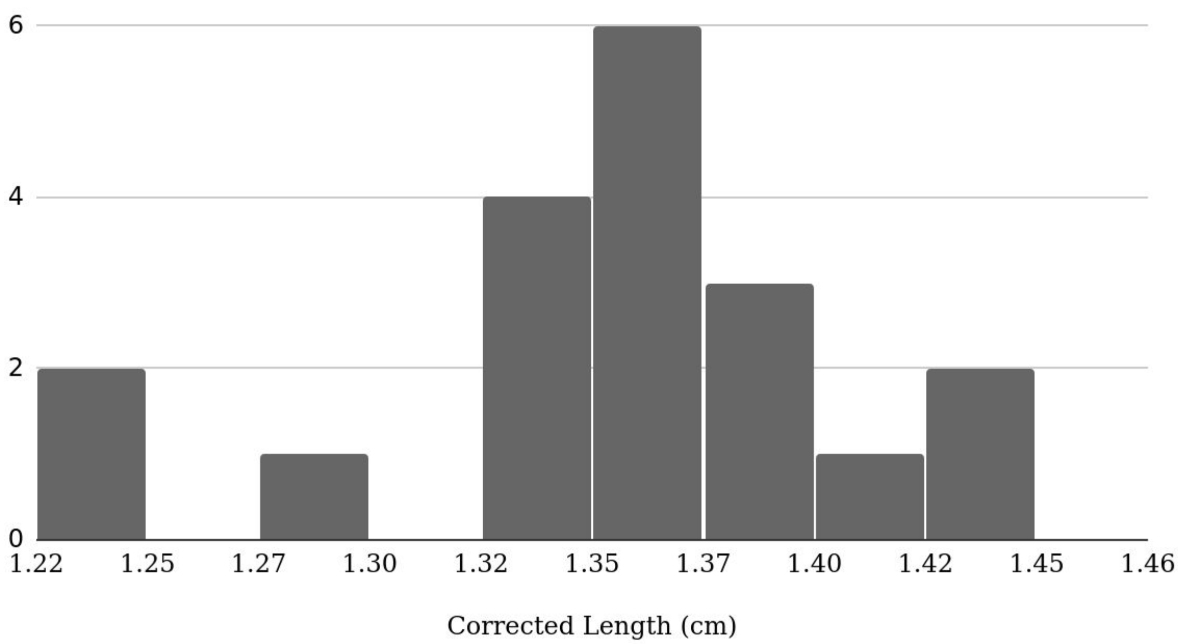


Figure 3. Histogram of length of acrylic pieces in bottle #5 (cm)



5. Conclusion

When calculating the volume from lengths and widths measured with the metric ruler and the vernier caliper in Part A, it was demonstrated that the lower random and instrumental errors calculated when using the vernier caliper generated a smaller error in the resulting volume by error propagation; a relative decrease of 94% uncertainty in the measuring instrument caused a relative decrease of 94.2% uncertainty in the volume calculation.

Densities calculated from the measurements of a digital balance, vernier caliper, and micrometer were close to the expected value. While the calculated value and random error for the cylindrical disk did not include the literature value for the known density of the alloy, its real difference from the density of the alloy was very small ($< 0.1 \frac{\mathrm{g}}{\mathrm{cm}^3}$), and the rectangular object's random error interval around the mean did capture the literature value.

Statistical measures were calculated on two larger samples of data, and a two-sample t-test for the difference of means produced a very small t value. As a result, the null hypothesis was not rejected, so there is no statistically significant difference between the means of the two samples.

A possible source of error was the quality of the geometries of the materials measured. The calculations assume that the geometries of the objects are a perfect cylinder (cylindrical disk) and perfect rectangular prisms (rectangular object and acrylic pieces), but this cannot be the case due to minor manufacturing errors. This was especially true of the acrylic piececs, of which some had very noticeable slants. It was also noted that there was a piece of paper taped to the center of the disk, which may make it slightly wider at the center and slightly more massive than if it were only the aluminum disk. Depending on the flaw in the geometry, each flaw may lead to systematically high, systematic low, or inconsistent (random) errors.

Another source of systematic error is the measurement of the diameter of the circular disk using the metric ruler. The method involved measuring the distance across a chort that is visually estimated to be a diameter without any sort of construction to verify it. Since the diameter is the longest chord, this will result in systematically low diameter measurements.

6. Questions

Question 1: Comparison of volume error values

The relative change in uncertainty (defined in (3.3, Change in uncertainty)) is -94% upon switching from the metric ruler to the vernier caliper. The relative change in uncertainty of the resulting volume calculation also is roughly -94%. A large decrease in uncertainty is expected, although the equal numerical value was unexpected — whether or not this is always the case is up to future experimentation and/or mathematical investigation.

Question 2: Agreement of density values

The calculated density values of $2.65\frac{\mathrm{g}}{\mathrm{cm}^3}$ for the rectangular prism and $2.6\frac{\mathrm{g}}{\mathrm{cm}^3}$ for the cylindrical disk were close to each other and the accepted literature value of $2.70\frac{\mathrm{g}}{\mathrm{cm}^3}$ (with 1.9% and 4% errors, respectively). It falls within the range of the random error for rectangular prism $\left(\pm 0.07\frac{\mathrm{g}}{\mathrm{cm}^3}\right)$; however, for the cylindrical disk, which has a lower mean that is coupled with a smaller error $\left(\pm 0.03\frac{\mathrm{g}}{\mathrm{cm}^3}\right)$, the literature value doesn't fall into its range of random error.

The fact that both errors were too low may imply a possible systematic error with measuring (see (Conclusion) for a more thorough discussion of this error). Nonetheless, all of these values are very close (within $0.1\frac{\mathrm{g}}{\mathrm{cm}^3}$), so the errors likely had a small effect.

Question 3: Shape of the sample histograms

Neither histogram of lengths of the sample is very normal-shaped. The sample for bottle #4 has a smaller spread than the sample for bottle #5. It is trimodal, but the peaks are closely packed near the center of the distribution. Bottle #5 is unimodal, but it has two gaps (which Bottle #4 doesn't have) and therefore two low outliers. Both distributions are mostly symmetric.

Even though neither distribution is perfectly unimodal and symmetric, they are mostly symmetric, do not have extremely skewed or distinct multimodal distributions, and are moderately-sized ($n = 20$). Thus, the distributions roughly satisfy the normality condition for the hypothesis test, and the two-sample t-test for the difference of means may be performed on this sample.

Question 4: Conclusions about the two-sample t-test

The t-statistic of 0.05009 is much less than the threshold $t < 1.96$ for a 5% significance level, which provides no evidence to support the alternative hypothesis that the difference of means is nonzero, and we fail to reject the null hypothesis that the difference in means is zero. This owes a large part to the means of both simulations only differing by less than a hundredth of a millimeter. The small STDOMs of both samples is also small ($\sigma \approx 0.01\mathrm{cm}$), which strengthens this claim by asserting that the means are already close to their true (population) values.