1 Purpose

The laws of reflection and refraction are observed and quantified in this lab. The main result of the measurements and calculations is the index of reflection of glass and tap water. Two methods are used, and their outputs are compared to the standard values within error bars.

In Part A, the calculation of the index of reflection of glass and tap water are tackled through an application of Snell's Law called Pfund's method, which is based on diffuse scattering and the critical refraction angle along the glass-air and glass-liquid boundaries. A petri dish is used as the glass sample, a laser used as the light source, and a paper with markings as the diffuse scatterer and the source of markings to measure; its thickness is measured with a micrometer, and the diameter of the boundaries of the rings caused by the scattering is measured indirectly with a vernier caliper. This also brings into light many possible sources of error, such as the method for indirect measurement.

In Part B, Snell's law is used more directly by measuring angles and using the Snell's law relationship to find the index of refraction of the liquid. This method explores using geometric construction to correctly measure the angles, as well as a different method of error propagation due to the fact that the angles cannot be simply averaged like the other measurement types.

2 Data

Table 1: Petri Dish Thickness

Sample	Measured thickness (cm)	Corrected thickness (cm)
1	0.2369	0.2354
2	0.2651	0.2636
3	0.2460	0.2445
4	0.2568	0.2553
5	0.2355	0.2340
6	0.2240	0.2225

Zero offset (cm)	0.0015
Instrumental error (cm)	0.0005
Random error (cm)	0.006
Mean (cm)	0.243
Uncertainty for the mean (cm)	0.006

Table 2: Ring diameter without liquid

Sample	Diameter (cm)
1	0.772
2	0.750
3	0.756
4	0.785
5	0.806
6	0.750

Zero offset (cm)	0.000
Instrumental error (cm)	0.002
Random error (cm)	0.009
Mean (cm)	0.770
Uncertainty for the mean (cm)	0.009

Table 3: Ring diameter with liquid

Sample	Diameter (cm)
1	1.814
2	1.850
3	1.950
4	1.884
5	1.834
6	1.840

Zero offset (cm)	0.000
Instrumental error (cm)	0.002
Random error (cm)	0.02
Mean (cm)	1.86
Uncertainty for the mean (cm)	0.02

Table 4: Snell's law method measured incident and refracted angles

Sample	Incident angle (deg)	Refracted angle (deg)
1	43.0	29.7
2	44.0	30.1
3	22.0	15.2
4	21.9	15.3
5	53.0	36.1
6	56.5	35.5

3 Calculations

Note that the full precision of each measurement is kept the final result of each calculation, in which the results are rounded for brevity.

3.1 Instrumental error

The vernier caliper has markings to the nearest 0.002cm, and reading it involves choosing the closest matching margin. Since there is no approximation between the markings, the instrumental uncertainty is $\delta S = \pm 0.002$cm.

The micrometer has markings to the nearest 0.001cm. Reading it involves visual estimation between markings, so the instrumental error for a reading is $\delta S = \pm 0.0005cm$. The micrometer had a nonzero zero-offset of 0.0015cm, so this value was subtracted from all of the measured values to yield the corrected values displayed in (Table 1).

The protractor has markings to the nearest 0.5°, so each angle reading has accuracy $\delta A_{read} = 0.25°$. Since each angle measurement is a function of two readings (the degree readings of the two rays that contain the angle), the error for a single angle measurement δA is calculated from the (independent) errors of the left and right readings.

$$\delta A = \sqrt{\delta A_{left}^2 + \delta A_{right}^2} = \sqrt{(0.25°)^2 + (0.25°)^2} = 0.354° = 0.4° \qquad (1)$$

3.2 Sample mean and random error

For each sample of length measurements (Part A), the measurement is recorded with both a "best value" and a specified random error. The "best value" is the mean $\bar{x}$, and the random error is reported as standard deviation of the mean (STDOM) σ_x.

For (Part B), calculating sample mean and standard deviation of the raw measured data was not applicable, as the angle values were not centered around a common point and did not have a common spread.

3.3 Calculation of n_g and error using Pfund's method

The calculation for the average index of refraction of glass, $\bar{n}_g$, is shown below in (2). This is derived in (Figure 2).

$$n_g = \frac{\sqrt{d^2 + 16t^2}}{d} \tag{2}$$

where d is the average diameter of the boundary between the gray and the bright ring, and t is the average thickness of the petri dish.

The error propagation for the calculation of n_g is shown below in (3). The values of d and t are dependent (i.e., a thicker petri dish would cause a change in ring diameter by the geometry of the problem; see (Figure 2)), and thus the absolute values of their errors are added. δd and δt are the larger of the random error and instrumental error for the respective metrics.

$$\frac{\partial n_g}{\partial d} = \frac{-16t^2}{d^2\sqrt{d^2 + 16t^2}}$$
$$\frac{\partial n_g}{\partial t} = -\frac{16t}{d\sqrt{d^2 + 16t^2}}$$
$$\delta n_g = \left|\frac{\partial n_g}{\partial d}\delta d\right| + \left|\frac{\partial n_g}{\partial t}\delta t\right| \tag{3}$$

Sample calculations

The calculation for n_g is shown below.

$$n_g = \frac{\sqrt{(0.770\text{cm})^2 + 16(0.243\text{cm})^2}}{0.770\text{cm}} = 1.61$$

The calculation for δn_g is shown below.

$$\delta n_g = \left|\frac{0.770\text{cm}^2 - 16(0.243\text{cm}^2)}{0.770\text{cm}^2\sqrt{0.770\text{cm}^2 + 16(0.243\text{cm})^2}}\,0.009\text{cm}\right| + \left|\frac{16(0.243\text{cm})}{0.770\text{cm}\sqrt{0.770\text{cm}^2 + 16(0.243\text{cm})^2}}\,0.006\text{cm}\right| = 0.0367$$

3.4 Calculation of n_l and error using Pfund's method

The calculation for the index of refraction of the liquid, n_l, is shown below in (4). This is derived from (Figure 3).

$$n_l = \frac{n_g\, d}{\sqrt{d^2 + 16t^2}} \tag{4}$$

where d is the diameter of the boundary between the gray and the bright stationary ring, and t is the thickness of the petri dish.

As before, d is dependent on t because of the geometry of the calculation. Since n_g is a function of t, it is also dependent on t. Thus, the absolute values of the errors are added. δd and δt are the larger of the random error and instrumental error for the respective metrics.

$$\frac{\partial n_l}{\partial n_g} = \frac{d}{\sqrt{d^2 + 16t^2}}$$

$$\frac{\partial n_l}{\partial d} = \frac{16 n_g t^2}{(d^2 + 16t^2)^{\frac{3}{2}}}$$

$$\frac{\partial n_l}{\partial t} = -\frac{16 n_g d t}{(d^2 + 16t^2)^{\frac{3}{2}}}$$

$$\delta n_l = \left|\frac{\partial n_l}{\partial n_g}\delta n_g\right| + \left|\frac{\partial n_l}{\partial d}\delta d\right| + \left|\frac{\partial n_l}{\partial t}\delta t\right| \tag{5}$$

Sample calculations

The calculation for n_l is shown below.

$$n_l = \frac{1.61(1.86\text{cm})}{\sqrt{1.86\text{cm}^2 + 16(0.243\text{cm}^2)}} = 1.431$$

The calculation for the error for n_l is shown below.

$$\delta n_l = \left|\frac{1.86\text{cm}}{\sqrt{1.86\text{cm}^2 + 16(0.243\text{cm}^2)}}0.0367\text{cm}\right| + \left|\frac{16(1.61)(0.243\text{cm})^2}{\sqrt{1.86^2 + 16(0.243\text{cm})}^3}0.02\text{cm}\right| + \left|-\frac{16(1.61)(1.86\text{cm}(0.243\text{cm}))}{\sqrt{1.86\text{cm}^2 + 16(0.243\text{cm})^2}^3}0.006\text{cm}\right| = 0.0436$$

3.5 Calculation of n_l and error using Snell's law method

Snell's law is stated in (6).

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{6}$$

In the case of this lab, let medium 1 (n_a, θ_a) be air, and medium 2 (n_l, θ_l) be the liquid. Approximating $n_a \approx 1$, then the index of refraction of the liquid can be rewritten in terms of the two angles:

$$n_l = \frac{\sin\theta_a}{\sin\theta_l} \tag{7}$$

Both the mean and error propagation are more difficult to calculate here than with the two methods from (Part A), since the raw measurements (angle values) cannot be averaged, and the mean and error calculations calculated from the means and errors of the measurements. This is because the angles are not centered around the same value, and do not have a similar spread – they vary widely between different samples. Therefore, the index of refraction of the liquid and its error have to be calculated for each sample, and then this set of data must be appropriately aggregated.

The aggregation for the mean is fairly intuitive: average the calculated n_l for each sample.

$$\bar{n}_l = \frac{1}{6}\sum_{i=1}^{6} n_{l_i} \tag{8}$$

For the error calculation, the error of a single sample can be calculated as shown in (9). The error calculation sums the absolute values of the errors, since the two angle values are dependent on one another, by Snell's law.

$$\frac{\partial n_l}{\partial \theta_i} = \frac{\cos\theta_i}{\sin\theta_r}$$
$$\frac{\partial n_l}{\partial \theta_r} = \frac{\sin\theta_i\cos\theta_r}{\sin^2\theta_r}$$
$$\delta n_l = \left|\frac{\partial n_l}{\partial \theta_i}\delta\theta_i\right| + \left|\frac{\partial n_l}{\partial \theta_r}\delta\theta_r\right| \tag{9}$$

where $\delta\theta_1$ and $\delta\theta_2$ are the instrumental error for the protractor measurements (see (1)). Since the mean is a function of six independent calculations, the error of the mean can be calculated by summing the errors of each calculation in quadriture, as shown in (10).

$$\delta\bar{n}_l = \frac{1}{6}\sqrt{\sum_{i=1}^{6}\left(\frac{\partial \bar{n}_l}{\partial n_{l_i}}\delta n_{l_i}\right)^2} = \frac{1}{6}\sqrt{\sum_{i=1}^{6}\delta n_{l_i}^2} \tag{10}$$

For all error calculations in this section, the angles are necessarily converted to their respective radian equivalents: The trigonometric and differential relations of sin and cos are derived from the radian interpretations of angles. Furthermore, using degrees will leave the error calculation with the wrong units (in degrees, rather than being dimensionless as n_l is) and will change the value of the error. The conversion from radians to degrees is shown in (**??**), and the converted angle values from (Table 4) are shown in (Table 5).

$$\theta_{\text{rad}} = \theta_{\circ} \times \frac{\pi}{180^\circ} \tag{11}$$

Sample calculations

An sample calculation for n_l (for the first sample of (Table 4)) is shown below.

$$n_{l_1} = \frac{\sin 0.750}{\sin 0.518} = 1.38$$

The calculation for the error for this sample, δn_l, is shown below.

$$\delta n_{l_1} = \left|\frac{\cos 0.750}{\sin 0.518}0.006\right| + \left|\frac{\sin 0.750\cos 0.518}{\sin^2 0.518}0.006\right| = 0.0145$$

Table 5: Angle values, converted from degrees to radians

Sample	Incident angle (rad)	Refracted angle (rad)
1	0.750	0.518
2	0.768	0.525
3	0.384	0.265
4	0.382	0.267
5	0.925	0.630
6	0.986	0.620

This calculation is repeated once for each sample. The calculated index of refraction and error for each sample are shown in (Table 6).

Table 6: Index of refraction of liquid and error for each sample using Snell's law

Sample	n_l	δn_l
1	1.376	0.0240
2	1.385	0.0236
3	1.429	0.0543
4	1.414	0.0536
5	1.355	0.0178
6	1.436	0.0183

The mean index of refraction is a simple arithmetic mean of the n_l values from (Table 6).

$$\bar{n}_l = \frac{1}{6}(1.376 + 1.385 + 1.429 + 1.414 + 1.355 + 1.436) = 1.399$$

The error calculation for the mean is shown below.

$$\delta n_l = \frac{1}{6}\sqrt{0.0240^2 + 0.0236^2 + 0.0543^2 + 0.0536^2 + 0.0178^2 + 0.0183^2} = 0.0145$$

4 Results

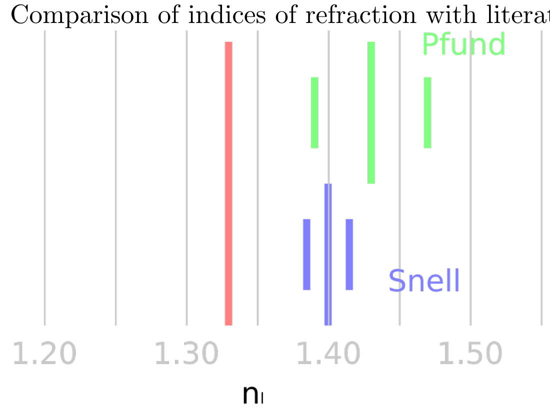
The calculated indices of refraction, along with their reported errors, are summarized in (Table 7), and a visual summary is provided in (Figure 1).

Table 7: Summary of indices of refraction

Material	Pfund's method	Snell's law method	Literature value
Liquid (tap water)	1.43 ± 0.04	1.40 ± 0.015	1.33[1]
Glass	1.61 ± 0.04	–	–

As can be seen in (Figure 1), neither the error of the mean for Pfund's method nor Snell's method captured the literature value.

Figure 1: Comparison of indices of refraction with literature value



[1] at 20°C, http://hyperphysics.phy-astr.gsu.edu/hbase/Tables/indrf.html

5 Conclusion

The calculated index of refraction using Pfund's method is 1.61 ± 0.04cm. While the exact composition of the petri dish is not known, the index of refraction of glasses is typically between 1.52 (crown glass) and 1.65 (heavy flint glass)[2], so it can be deduced that this calculated value is reasonable.

The calculated index of refraction of the liquid (tap water) was 1.43 ± 0.04 using Pfund's method, and 1.40 ± 0.015 using the Snell's law method. The literature value is 1.33. Neither of the calculated values' error intervals capture the literature value – however, the two calculated error intervals are small (0.04 and 0.015) and overlap, which may suggest that some part of the procedure caused a systematically high calculation for indices for refraction.

There were many sources of error, delineated below.

For Pfund's method, the index of refraction of air was approximated to be 1 to make calculations simpler. However, the literature value of air at STP is 1.00029[2]. Since, in the calculation for critical angle of the glass, $n_2 = \frac{n_a}{\sin \theta_c}$, this would cause the resulting answer to be systematically lower but a small factor ($\frac{1}{1.00029}$ of the value obtained using the literature value). This likely had a very small factor, judging by the fact that the results were mostly reported to only three significant figures.

The procedure for Pfund's method asked for many indirect measurements, and this was likely the source of the largest (random and overall) error in the results since the measurement tools are very precise, even if not quantifiable and thus not factored into the error propagation. The procedure asks to visually determine the diameter of a ring, by visually "copying" the ring onto a second sheet of paper, which was then measured by a Vernier caliper. It is not certain if this error is random or systematic, because it cannot be quantified – there is no way to check what the true diameter of the rings are. Adding to the difficulty of the visual determination of ring diameter is that sometimes the edge of the rings were blurry, and the paper was broken at some places, causing the reflection and the edge of the ring to be less strongly-defined in those sections.

In Pfund's method, the thickness of the bottom of the petri dish is measured. However, while the model assumes that the thickness is only of glass, the measurement obtained with the micrometer caliper also included a layer of paper, and in some places, tape. This makes the value of t systematically higher than the true value, which in turn makes n_g systematically higher (but has no effect on n_l, since it cancels out).

In the Snell's law procedure, there is much random error introduced by the procedure. Firstly, a circle had to be drawn around the petri dish such that the edge of the petri dish was right above the circle, but small deviations in pencil angle when drawing the circle could make the circle larger or smaller than the actual petri dish diameter. There is some random error in pin placement and the construction of the center of the circle (using perpendicular bisectors) that wasn't accounted for, because it doesn't quantifiably affect any measurement

[2]http://hyperphysics.phy-astr.gsu.edu/hbase/Tables/indrf.html

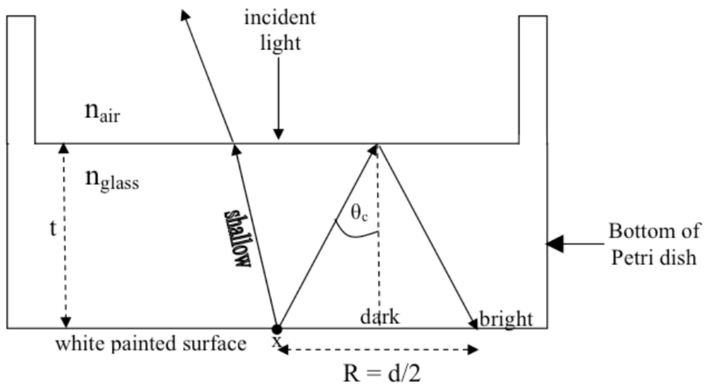
(but it does affect the angle measurements with the protractor).

For both methods, the tap water seemed to be a little cloudy (have some particulate matter). This may have introduced some systematic error in the true n_l value, making it deviate from the literature value. This may help explain the high precision but low accuracy (relative to the literature value) for both n_l calculations.

6 Answers to questions

6.1 Derivation of n_g formula for Pfund's method

Figure 2: Schematic of setup to determine n_g



We begin with Snell's law:

$$n_a \sin\theta_a = n_g \sin\theta_g$$

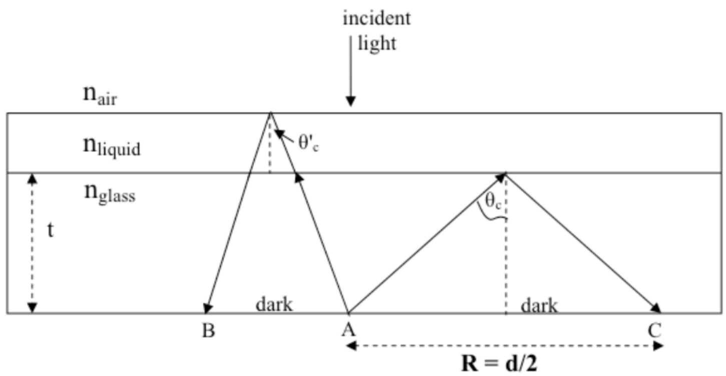
We assume that $n_a \approx 1$. At the critical angle in the glass θ_{g_c}, $\theta_a = 90°$. The Snell's law equation becomes

$$n_g = \frac{(1)\sin 90°}{\sin\theta_{g_c}} = \csc\theta_{g_c} = \frac{\sqrt{((d/2)/2)^2 + t^2}}{(d/2)/2} = \frac{\sqrt{d^2 + 16t^2}}{d}$$

(The calculation of csc comes from the right-triangle geometry in the schematic.)

6.2 Derivation of n_l formula for Pfund's method

Figure 3: Schematic of setup to determine n_l



The bright ring caused by the reflection on the left is caused by the internal reflection by the liquid-air boundary. We are interested instead on the reflection on the right, which uses the glass-liquid boundary. Again, we begin with Snell's law.

$$n_g \sin\theta_g = n_l \sin\theta_l$$

At the critical angle for this boundary from the glass n_{g_c}, we get $\theta_g = 90°$ Solving for n_l, we get:

$$n_l \sin 90° = n_g \sin\theta_{g_c}$$

The rest of the derivation follows similarly for that previously shown for n_g:

$$n_l = n_g \frac{(d/2)/2}{\sqrt{((d/2)/2)^2 + t^2}} = \frac{n_g d}{4\sqrt{(d/4)^2 + t^2}} = \frac{n_g d}{\sqrt{d^2 + 16t^2}}$$

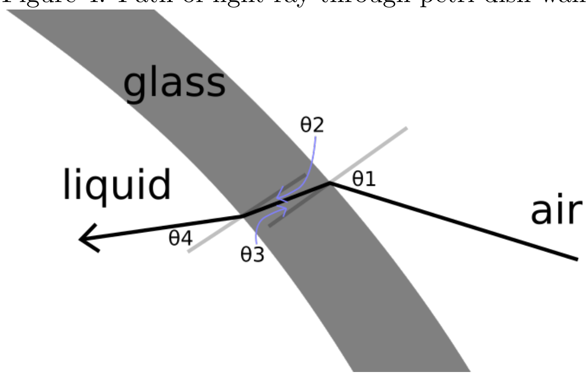
(This again uses the right-angle geometry of the schematic.)

6.3 Assumptions made by the Snell's law technique

The Snell's law technique approximated the refraction to be between air and liquid on the edge of the petri dish, assuming that the refraction caused by the petri dish glass is trivial. This approximation can be justified by the fact that the light beam passing through the petri dish passes through a thin layer of glass, with two nearly-parallel refracting surfaces and two refractions in opposite directions.

6.4 Derivation of n_l formula for Pfund's method

Figure 4: Path of light ray through petri dish wall



There are two Snell's law relevant to the setup in (Figure 4):

$$n_a \sin\theta_1 = n_g \sin\theta_2$$

$$n_g \sin\theta_3 = n_l \sin\theta_4$$

Note that, while the petri dish is curved, if you zoom in on a small region, the inner and outer edge are essentially parallel, thus $\theta_3 \approx \theta_4$, so

$$n_a \sin\theta_1 \approx n_l \sin\theta_4$$

This demonstrates that the glass's refraction is small and can be approximated away by assuming that the glass is thin and not overly curved. From this diagram it can also be intuited that the effects of this are that the two inner angles within the glass are slightly different and, since there are two points of refraction, the path of light may be offset a little bit from if there was only an air-liquid boundary. This approximation is reasonable by the justification above, and by the small error of the mean for the Snell's law calculations of 0.015.

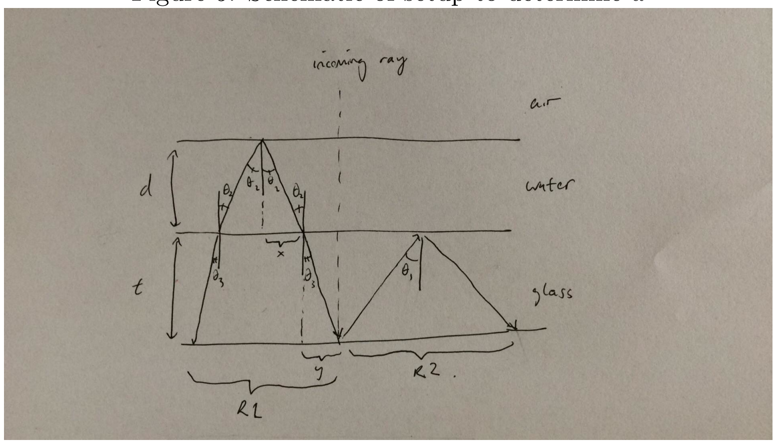
As with the Pfund's law method, the Snell's law technique also assumes that the liquid is transparent or translucent, and that $n_a = 0$. These are reasonable, given that the liquid in question is transparent and the index of refraction of air is known to be very close to 1.

6.5 Will Pfund's method work for liquids of all n?

Pfund's method used here to calculate n_l is not applicable for liquids of all n. It uses the property of total internal reflection of the glass, and total internal reflection only occurs if the liquid has a lower n than glass. In other words, the critical angle (only after which exists the phenomenon of total internal reflection) only exists if the refracted angle is shallower from the surface than the incident angle, which happens only if the index of refraction of the second medium is lower than that of the first.

6.6 Extra credit: depth of water

Figure 5: Schematic of setup to determine d



Let R_1 be the radius of the ring caused by the total internal reflection of water with air, R_2 be the radius of the ring caused by total internal reflection of glass with water (the quantity measured in Pfund's method for n_l).

For θ_2, which is the critical angle of water with air:

$$n_l \sin\theta_2 = n_a \sin 90^\circ = 1 \tag{12}$$

For θ_1, which is the refracted angle of the light ray from liquid to glass with incident angle θ_2:

$$n_g \sin\theta_1 = n_l \sin\theta_2 = 1 \tag{13}$$

By trigonometry:

$$\sin\theta_1 = \frac{y}{\sqrt{y^2+t^2}} \tag{14}$$

$$\sin\theta_2 = \frac{x}{\sqrt{x^2+d^2}} \tag{15}$$

By (13), (14):

$$n_g \frac{y}{\sqrt{y^2+t^2}} = 1$$

$$n_g^2 y^2 = y^2 + t^2$$

$$y = \frac{t}{\sqrt{n_g^2 - 1}} \tag{16}$$

Similarly, by (13), (15):

$$x = \frac{d}{\sqrt{n_l^2 - 1}} \tag{17}$$

By definition of x, y in the diagram, $2(x+y)=R_1$. Thus:

$$2\left(\frac{d}{\sqrt{n_l^2-1}}+\frac{t}{\sqrt{n_g^2-1}}\right)=R_2 \tag{18}$$

Solving for d:

$$d=\left(R_2-\frac{2t}{\sqrt{n_g^2-1}}\right)\frac{\sqrt{n_l^2-1}}{2} \tag{19}$$