Digital signature
Name: Jonathan Lam
Partner's name: Andrew Kim
Date: 12/09/19

1 Purpose

Optical diffraction is used to estimate the width of a human hair. This involved the calculation of the wavelength of a light source (a laser) by measuring attributes of an N-slit diffraction spectrum. This calculated wavelength is used to estimate the human hair's width by measuring attributes of a wide single-slit diffraction pattern.

2 Data

Note 1: The position of the front of the laser was measured, not the position of the diffraction grating, single slit, or hair. However, the objects or slits were put with a ring stand and clamp as close as possible to the laser (≈1 to 3mm) in front of the laser to reduce this error.

Note 2: The reported instrumental errors are those of a single reading, before any calculations; e.g., the instrumental error for the optical bench is not equal to the error for a length measurement (which involves two readings, and whose error is taken into account in the error propagation sections).

Note 3: (Table 5) and (Table 8) are identical, as the setup was not changed between these parts (except that the Vernier caliper was replaced with a hair). This also shows the flexibility of this setup to determine either wavelength or slit width, as well as the fascinating result that an object and a slit of the same thin finite width can generate the same diffraction pattern.

Note 4: The wavelength of the lasers used in this experiment have wavelengths known to be approximately 650nm, but the exact value is not known (i.e., there was no sticker on our laser indicating its wavelength),

2.1 Part A. N-slit diffraction pattern

Table 1: Bench measurements

Pos. screen (cm)	121.82
Pos. laser (cm)	105.03
Laser offset (cm)	1.06
Dist. laser to screen (cm)	15.73
Instrumental error (cm)	0.05

Table 2: Distances from center to intensity maxima

Diff. grating line density (lines/mm)	1000
Dist. center to left maximum (cm)	12.999
Dist. center to right maximum (cm)	13.548
Instrumental error (cm)	0.002

2.2 Part B. (Wide) single-slit diffraction pattern

Table 3: Measured distances from fourth minima to the left to other minima

a (cm)	$d_{-4,4}$ (cm)	$d_{-4,3}$ (cm)	$d_{-4,2}$ (cm)	$d_{-4,1}$ (cm)	$d_{-4,-1}$ (cm)	$d_{-4,-2}$ (cm)	$d_{-4,-3}$ (cm)
0.02	2.458	2.236	1.962	1.610	0.920	0.638	0.300
0.03	2.080	1.810	1.634	1.300	0.796	0.628	0.300
0.04	1.472	1.298	1.098	0.942	0.648	0.414	0.178

Table 4: Distances of minima to center of diffraction pattern (h)

a (cm)	h (cm)							
	$p = 4$		$p = 3$		$p = 2$		$p = 1$	
	left	right	left	right	left	right	left	right
0.02	1.283	1.265	0.971	0.965	0.697	0.627	0.345	0.345
0.03	1.032	1.048	0.762	0.748	0.586	0.420	0.252	0.252
0.04	0.677	0.795	0.503	0.617	0.303	0.381	0.147	0.147

Table 5: Distance from laser to screen

Pos. screen (cm)	124.70
Pos. laser (cm)	8.65
Laser offset (cm)	1.06
Dist. laser to screen (cm)	114.99
Instrumental error (cm)	0.05

2.3 Part C. Hair diffraction pattern

Table 6: Measured distances from fourth minima to the left to other minima

Owner	$d_{-4,4}$ (cm)	$d_{-4,3}$ (cm)	$d_{-4,2}$ (cm)	$d_{-4,1}$ (cm)	$d_{-4,-1}$ (cm)	$d_{-4,-2}$ (cm)	$d_{-4,-3}$ (cm)
Andrew	8.748	7.750	6.792	5.450	3.422	2.298	1.780
Jon	7.278	6.300	5.300	4.412	2.704	1.830	0.914

Table 7: Distances of minima to center of diffraction pattern (h)

Owner	h (cm)							
	$p = 4$		$p = 3$		$p = 2$		$p = 1$	
	left	right	left	right	left	right	left	right
Andrew	4.312	4.436	3.314	2.656	2.356	2.138	1.014	1.014
Jon	3.720	3.558	2.742	2.644	1.742	1.728	0.854	0.854

Table 8: Distance from laser to screen

Pos. screen (cm)	124.70
Pos. laser (cm)	8.65
Laser offset (cm)	1.06
Dist. laser to screen (cm)	114.99
Instrumental error (cm)	0.05

3 Explanation of errors

If the backplane holding the paper is not normal (i.e., perpendicular) to the incident rays, it may cause asymmetry and random error in the measurements (it will systematically increase the values of h on one side and decrease the values of h on the other). We didn't realize or attempt to quantify or correct this source of error. While there is a large deviation from the two intensity maxima from the center in Part A, it is less clear from Part B whether the backplane was tilted, as the measurements from one side are not notably systematically higher or lower than the other.

There is also the possibility of systematic error from errors with measuring offsets. We did not measure the offset of the paper screen from the center of the backplane (thus considering the paper to have zero offset from its holding device), but this may cause l to be systematically small. Similarly, we estimated the distance between the slit(s) or object and the paper as the distance between the tip of the laser and the paper, and placed the slit(s) or object very close to the tip of the laser, and this may cause l to be systematically large. However, these offsets (on the order of roughly 1-3mm) are deemed insignificant compared to the distance l (on the order of dozens of centimeters). To further mitigate this relative error, we made l large (15.73cm for Part A and 114.99cm for Parts B and C) by increasing the distance between the laser and the screen.

There is some random error introduced by marking the position of the intensity maxima (in Part A) or minima (in Parts B and C) on the paper. For either intensity extrema, it is difficult to pinpoint the exact position of the extrema, even if the bright or dark spots are fairly small (an estimate of this visual error may be ± 0.5mm). Furthermore, the intensity minima is not exactly at the center of the bright or dark regions because of each extrema's asymmetry (except the central peak); however, this is likely dwarfed by visual and instrumental uncertainties.

It was difficult to get correct measurements with the calipers, as we were measuring between penciled lines on paper and it is difficult to measure the perpendicular distance because of the nature of the calipers. This should add some small, unquantifiable random error to the caliper's instrumental uncertainty.

4 Calculations

4.1 Instrumental and length errors

Two measurement tools were used: the optical bench and a Vernier caliper. The instrumental error for lengths obtained using the Vernier caliper is 0.002cm. (See the note in the Explanation of errors section about additional error introduced when using the Vernier calipers.)

The instrumental error for the optical bench is 0.05cm. For lengths obtained using the optical bench, the error is 0.07cm, because lengths are obtained by the difference of two independent measurements with an instrumental error of 0.05cm each, i.e., for a length measurement s obtained using the optical bench,

$$s = s_2 - s_1$$

$$\delta s = \sqrt{\delta s_2^2 + \delta s_1^2} = \sqrt{0.05\text{cm}^2 + 0.05\text{cm}^2} = 0.07\text{cm}$$

4.2 Mean and standard deviation of the mean (STDOM)

The best for a variable x is calculated is given by the mean.

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

The error propagation for the mean, assuming values of x are independent, is given by the RMS of the errors.

$$\delta\bar{x} = \frac{1}{N}\sqrt{\sum_{i=1}^{N}\delta x_i^2}$$

The STDOM is another interpretation of the error of the mean and is given by the following equation.

$$\sigma_{\bar{x}} = \frac{1}{\sqrt{N}}\left(\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x}^2)\right)$$

Where applicable (i.e., in Parts B and C), both the error propagation for the mean and STDOM are calculated, and the larger error is reported and/or used for aggregated error propagation calculations, i.e.:

$$x = \bar{x} \pm \max(\delta\bar{x}, \sigma_{\bar{x}})$$

4.3 Calculation of wavelength from diffraction grating

In section 7.1, we derive (Eq. 8) for the positions of the intensity maxima for a laser's light from an N-slit grating.

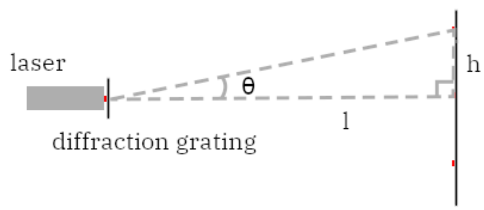
$$\sin\theta = \frac{m\lambda}{d}$$

where d is the slit density, and θ is the angle from the optical (horizontal) axis to the m-th intensity maximum on either side. In particular, since only one intensity maxima region was viewed on either side, we have two samples for the $m = 1$ case, with the same d value and differing θs. Rearranging the equation, we solve for λ.

$$\lambda = d \sin\theta$$

in the case where $m = 1$. $\sin\theta$ is calculated from the ratio of lengths rather than calculating the sine of a measured angle, as can be viewed in (Fig. 1).

Figure 1: Setup for estimation of wavelength from diffraction grating



Thus

$$\sin\theta = \frac{h}{\sqrt{l^2 + h^2}} \tag{1}$$

4.3.1 Calculation of l

The calculation of the distance l from laser tip (which is placed roughly a millimeter away from the diffraction grating) to screen is simply the difference of the position of the screen and laser positions, plus the offset of the tip of the laser from the center of the reading. In other words, let p_s be the screen position, p_l be the laser position, and o_l be the laser offset from the reading be from (Table 1); then

$$l = p_s - (p_l - o_l) \tag{2}$$

and thus

$$\delta l = \sqrt{\delta p_s^2 + \delta p_l^2 + \delta o_l^2}$$

Since p_s and p_l were estimated from the optical bench (instrumental uncertainty 0.05cm) and o_l was estimated with the Vernier caliper (instrumental uncertainty 0.002cm), the error δl is

$$\delta l = \sqrt{(0.05\text{cm})^2 + (0.05\text{cm})^2 + (0.002\text{cm})^2} = 0.07\text{cm} \tag{3}$$

4.3.2 Error propagation for wavelength from diffraction grating

We first calculate $\delta(\sin\theta)$, which will also be used in error calculations for subsequent sections.

$$\delta(\sin\theta) = \sqrt{\left(\frac{\partial(\sin\theta)}{\partial h}\delta h\right)^2 + \left(\frac{\partial(\sin\theta)}{\partial l}\delta l\right)^2} \tag{4}$$

$$= \sqrt{\left(\frac{l^2\delta h}{\sqrt{l^2+h^2}^3}\right)^2 + \left(\frac{hl\delta l}{\sqrt{l^2+h^2}^3}\right)^2} = \sqrt{\frac{(l^2\delta h)^2 + (hl\delta l)^2}{(l^2+h^2)^3}}$$

We don't perform an error propagation on d, so the error propagation for λ is straightforward.

$$\delta\lambda = d\delta(\sin\theta)$$

Another interpretation for uncertainty is half of the range of the two wavelengths.

$$\delta\lambda = \frac{|\lambda_1 - \lambda_2|}{2}$$

4.3.3 Sample calculations

Using (Table 1), we obtain l:

$$l = 121.82\text{cm} - (105.03\text{cm} + 1.06\text{cm}) = 15.73\text{cm}$$

Now we calculate $\sin\theta$ and λ, using the above calculated results and (Table 1).

$$\sin\theta = \frac{12.999\text{cm}}{\sqrt{(12.999\text{cm})^2 + (15.73\text{cm})^2}} = 0.637$$

$$\lambda = (0.001\text{mm})(0.637) = 637\text{nm}$$

Now we calculate error propagation. δh is the instrumental error for the Vernier caliper, and δl is as calculated in (Eq. 3).

$$\delta(\sin\theta) = \sqrt{\frac{((15.73\text{cm})^2(0.002\text{cm}))^2 + ((15.73\text{cm})(12.999\text{cm})(0.07\text{cm}))^2}{((15.73\text{cm})^2 + (12.999\text{cm})^2)^3}}$$

$$= 0.002$$

$$\delta\lambda = (0.001\text{mm})(0.002) = 2\text{nm}$$

The average value of the two calculated wavelengths and the uncertainty (half of the range) are displayed below.

$$\bar{\lambda} = \frac{637 - 653}{2} = 645\text{nm}$$

$$\delta\bar{\lambda} = \frac{653 - 637}{2} = 8\text{nm}$$

This value is much larger than the calculated error propagation uncertainties of $\bar{1}.7$nm and $\bar{1}.4$nm (respectively) of the two trials, which hints that there is some larger, unaccounted-for error in the error propagation (see explanation of errors section, particularly about the screen's slant).

4.4 Calculation of wavelength from single slit

In section 7.2, we derive (Eq. 10) for the positions of the intensity minima of a laser's light from a single finite-width slit.

$$\sin\theta = \frac{p\lambda}{a}$$

where a is the width of the slit and θ is the angle from the optical (horizontal) axis to the p-th minima. The equation is rearranged to solve for wavelength.

$$\lambda = \frac{a}{p}\sin\theta$$

where $\sin\theta$ is calculated as in (Eq. 1).

4.4.1 Calculation of l and h

l is calculated the same way as in (Eq. 2).

Measurements for h use measurements from (Table 3) to obtain the values in (Table 4). All length measurements were taken with respect to the fourth minimum on the left side (i.e., length from fourth to third minima, length from fourth to second minima, ..., length from fourth to fourth minima on opposite sides); therefore, seven length measurements were taken for each aperture width. The distance from the fourth minimum to the center was approximated as halfway between the first minima on each side of the center, and each value for h is the difference between the length of the fourth minimum and the desired minimum and the length between the fourth minimum and the center. For clarity, denote the minima on the left to have negative indices (i.e., minima -1, -2, -3, -4) and the minima on the right to have positive indices (i.e., minima 1, 2, 3, 4), denote the distance between the i-th and j-th minima as $d_{i,j}$, and call the center the minimum with index 0. Thus:

$$d_{-4,0} = \frac{d_{-4,-1} + d_{-4,1}}{2}$$

and

$$h_i = d_{i,0} = |d_{-4,i} - d_{-4,0}|$$

Since all lengths were measured with respect to minimum -4,

$$\delta d_{-4,i} = 0.002\text{cm}\ \forall i \in -4, -3, \ldots, 4$$

and thus

$$\delta h = \sqrt{(0.002\text{cm})^2 + \left(\frac{0.002\text{cm}}{2}\right)^2 + \left(\frac{0.002cm}{2}\right)^2} = 0.00\bar{2}4\text{cm} \tag{5}$$

4.4.2 Error propagation for calculation of wavelength from single slit

This calculation is very similar to that of the previous section, except now a is also a measured value with an uncertainty. Since measurement error in a affects the error of h, a and θ are dependent. Embedding the error expression for $\delta(\sin\theta)$ from (Eq. 4), we have

$$\delta\lambda = \left|\frac{\partial\lambda}{\partial a}\delta a\right| + \left|\frac{\partial\lambda}{\partial(\sin\theta)}\delta(\sin\theta)\right| = \left|\frac{(\sin\theta)\delta a}{p}\right| + \left|\frac{a\delta(\sin\theta)}{p}\right|$$

Since all of the terms will be positive, this simplifies to

$$\delta\lambda = p^{-1}((\sin\theta)\delta a + a\delta(\sin\theta)) \tag{6}$$

4.4.3 Sample calculation

The sample calculation shown is that of the fourth minimum on the right. We calculate h, l, and $\sin\theta$, and λ, for $a = 0.002$cm in that order.

$$h = \left|2.548\text{cm} - \frac{1.962\text{cm} + 1.610\text{cm}}{2}\right| = 1.283\text{cm}$$

$$l = 124.70\text{cm} - (8.65\text{cm} + 1.06\text{cm}) = 114.99\text{cm}$$

$$\sin\theta = \frac{1.283\text{cm}}{\sqrt{(1.283\text{cm})^2 + (114.99\text{cm})^2}} = 0.0112$$

$$\lambda = \frac{(0.02\text{cm})(0.0112)}{4} = 558\text{nm}$$

We calculate the error next. δh is given by (Eq. 5), and δl is given by (Eq. 3). δa is the instrumental error of the Vernier caliper.

$$\delta(\sin\theta) = \sqrt{\frac{((114.99\text{cm})^2(0.002\text{cm}))^2 + ((1.283\text{cm})(114.99\text{cm}))^{2^{\,3}}}{((1.283\text{cm})^2 + (114.99\text{cm})^2)}} = 0.0000\bar{2}24$$

$$\delta\lambda = \frac{(0.0112)(0.002\text{cm}) + (0.02\text{cm})(0.0000224)}{4} = \bar{5}6.9\text{nm}$$

The mean, standard deviation of the mean (STDOM), and error propagation for the mean is taken for each aperture size. For $a = 0.02$cm, the calculations are shown below.

$$\bar{\lambda}_{a=0.02\text{cm}} = \frac{558\text{nm} + 563\text{nm} + \cdots + 550.\text{nm}}{8} = 573\text{nm}$$

$$\sigma_{\bar{\lambda}_{a=0.02}\text{cm}} = \frac{\sqrt{(558\text{nm} - 573\text{nm})^2 + (563\text{nm} - 573\text{nm})^2 + \cdots + (550.\text{nm} - 573\text{nm})^2}}{(8-1)\sqrt{8}}$$
$$= \bar{8}.83\text{nm}$$
$$\delta_{\bar{\lambda}_{a=0.02}\text{cm}} = \frac{\sqrt{(57\text{nm})^2 + (58\text{nm})^2 + \cdots + (56\text{nm})^2}}{8} = \bar{2}1\text{nm}$$

Thus $\bar{\lambda}_{a=0.02}\text{cm} = 570 \pm 20\text{nm}$.

Next, the best value from all aperture sizes is aggregated by taking the mean of the calculated $\bar{\lambda}$ from each aperture size, and the error calculation is calculated likewise (calculate STDOM and error propagation for the mean and reporting the larger error). The calculations are similar to those shown above for each aperture size and thus omitted. The results for this section are summarized in (Table 10).

4.5 Calculation of width of human hair

Optical diffraction predicts that the diffraction pattern for a thin finite-width gap is the same as that produced by a thin object of the same finite width. (Eq. 10) is used like in the preceding section, and manipulated to solve for the a, the width of the object.

$$a = \frac{p\lambda}{(\sin\theta)}$$

where $\sin\theta$ is calculated as in (Eq. 1).

4.5.1 Error propagation for the width of human hair

The errors $\delta(\sin\theta)$ (calculated in (Eq. 4)) and $\delta\lambda$ (calculated in (Eq. 6)) are independent and thus added in quadrature.

$$\delta a = \sqrt{\left(\frac{\partial a}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial a}{\partial(\sin\theta)}\delta(\sin\theta)\right)^2} = \sqrt{\left(\frac{p\delta\lambda}{(\sin\theta)}\right)^2 + \left(\frac{p\lambda\delta(\sin\theta)}{(\sin\theta)^2}\right)^2}$$
$$= \frac{p}{(\sin\theta)^2}\sqrt{((\sin\theta)\delta\lambda)^2 + (\lambda\delta(\sin\theta))^2}$$

4.5.2 Sample calculation

The sample calculation shown is that for Andrew's hair using the fourth minimum on the right. The process is very similar to the previous sample calculation, and it uses the best value for λ calculated in the previous section.

$$h = \left|8.748\text{cm} - \frac{5.450\text{cm} + 3.422\text{cm}}{2}\right| = 4.312\text{cm}$$
$$l = 124.70\text{cm} - (8.65\text{cm} + 1.06\text{cm}) = 114.99\text{cm}$$
$$\sin\theta = \frac{4.312\text{cm}}{\sqrt{(4.312\text{cm})^2 + (114.99\text{cm})^2}} = 0.0375$$

$$a = \frac{4(611\text{nm})}{0.0375} = 65.2\mu\text{m}$$

For the error propagation, δh and δl are known, as in the previous section. $\delta\lambda$ is the error for λ calculated in the previous section.

$$\delta(\sin\theta) = \sqrt{\frac{((114.99\text{cm})^2(0.002\text{cm}))^2 + ((114.99\text{cm})(4.312\text{cm})(0.07\text{cm}))^2}{((114.99\text{cm})^2 + (4.312\text{cm})^2)^3}} = 0.0000\bar{3}13$$

$$\delta a = \frac{4}{0.0375^2}\sqrt{(0.0375 * (14\text{nm}))^2 + ((611\text{nm}) * 0.0000237)} = 1.48\mu\text{m}$$

The mean, STDOM, and error propagation for the mean are identical calculations as those in the preceding section for each aperture size, so no sample calculations will be shown here. The results from this section are summarized in (Table 11).

5 Results

5.1 Part A

Table 9: Calculation of wavelength from N-slit diffraction

Sample 1 λ (nm)	Sample 2 λ (nm)	$\bar{\lambda}$ (nm)
637	653	645 ± 8

5.2 Part B

Table 10: Wavelength calculated from single-slit diffraction

a (cm)	$\bar{\lambda}$ (nm)	$\delta\bar{\lambda}$ (err. prop.) (nm)	$\delta\bar{\lambda}$ (STDOM) (nm)
0.02	570	20	9
0.03	660	20	20
0.04	$6\bar{0}0$	12	30

Reported $\bar{\lambda}$ (nm)	610 ± 14

5.3 Part C

Table 11: Width of human hair calculated by single-slit diffraction

Owner	$\bar{a}$ (μm)	$\delta\bar{a}$ (err. prop.) (μm)	$\delta\bar{a}$ (STDOM) (μm)	Reported $\bar{a}$ (μm)
Andrew	66.9	0.5	2	67 ± 2
Jon	79.8	0.7	0.9	79.8 ± 0.9

6 Conclusion

Reasonable values were obtained from Parts A and B for the wavelength of the laser, 645 ± 8nm and 610 ± 14nm, respectively. While the latter doesn't capture the expected approximate true wavelength of roughly 650nm within one standard deviation, the 6% error is small.

In Part B, the diameter of Andrew's and Jon's hairs were determined to be $67 \pm 2\mu$m and $79.8 \pm 0.2\mu$m, respectively. These are reasonable values; Brian Ley from *The Physics Factbook* estimates that most human hair falls in the range of 17 to 181μm[1]. The small standard deviation and the small error of the slit method in Part B indicates that this method has a high precision for very thin objects, most much higher than can be achieved with Vernier calipers, linear scales, or more coarse instruments (and perhaps similar to or better than micrometer calipers).

[1] https://hypertextbook.com/facts/1999/BrianLey.shtml

7 Answers to questions

7.1 Intensity maxima in N-slit diffraction

The intensity pattern of an N-slit pattern, with slits spaced d distance away on a distant screen is given by (Eq. 7), as defined in class.

$$I(\delta) = I_0 \left[\frac{\sin \frac{N\delta}{2}}{\sin \frac{\delta}{2}} \right]^2 \tag{7}$$

where $\delta = \frac{2\pi}{\lambda} d \sin\theta$ (the phase delay) and θ is the angle between the ray pointing to the position of interest on the screen and the central ray. While it seems plausible that the maxima lie elsewhere, the maxima lie in the most obvious positions: where the denominator of the fraction is zero. Thus $x = \pi m,\ m = \pm 1, \pm 2, \ldots$.

$$\frac{\delta}{2} = m\pi \Rightarrow \frac{2\pi}{\lambda} d \sin\theta = 2\pi m \Rightarrow \sin\theta = \frac{m\lambda}{d} \tag{8}$$

We may calculate the maximum intensity at the points. Let $x = \frac{\delta}{2}$. We use L'Hopital's rule twice to evaluate this limit.

$$I\text{max} = \lim_{x\to 0} I_0 \left[\frac{\sin Nx}{\sin x} \right]^2 = I_0 \lim_{x\to 0} \frac{2N\sin(Nx)\cos(Nx)}{2\sin(x)\cos(x)} = I_0 \frac{N\sin(2Nx)}{\sin(2x)}$$

$$= I_0 N \lim_{x\to 0} \frac{2N\cos 2Nx}{2\cos 2x} = I_0 N^2 \frac{cos(0)}{cos(0)} = I_0 N^2$$

7.2 Intensity minima in single-slit diffraction

The intensity pattern of a (finite-width) single-slit diffraction, using the same variable conventions as in the previous section, is given by (Eq. 9).

$$I(\beta) = I_0 \left(\frac{\sin\beta}{\beta} \right)^2 \tag{9}$$

where $\beta = \frac{\pi}{\lambda} a \sin\theta$. This clearly has minima when $\sin\beta = 0$, or, equivalent, when $\beta = p\pi,\ p = \pm 1, \pm 2, \ldots$

$$\frac{\pi}{\lambda} a \sin\theta = p\pi \Rightarrow \sin\theta = \frac{p\lambda}{a} \tag{10}$$