MA326 Linear Algebra – Definitions, Theorems, and other Results

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November 11, 2019

Contents

1 Appendix C: Fields

DEF binary operation A binary operation is a mapping of two elements of the same set to the same set, i.e., $\text{op}_{\text{bin}}(x_1, x_2) : S \times S \to S$.

DEF field A field F is a set on which the binary operators addition $(+)$ and multiplication $(\cdot)$ are defined, such that addition and multiplication are closed on the set and the five field axioms apply:

1. Commutativity of addition and multiplication
2. Associativity of addition and multiplication
3. Existence of (distinct) identity elements for addition (0) and multiplication (1)
4. Existence of inverse elements for addition and multiplication (except with 0 for multiplication)
5. Distributivity of multiplication over addition

THM C.1 (Cancellation Laws for Fields) For arbitrary elements $a, b \in F$, the following statements are true:

- $a + b = c + b \Rightarrow a = c$
- $a \cdot b = c \cdot b \Rightarrow a = c$

THM C.1 COR 1: Uniqueness of identity and inverse elements (Proof straightforward)

DEN: additive and multiplicative inverse $\forall a \in F$, a's additive and multiplicative inverses exist (F4) and are unique (THM C.1 COR 1), and are denoted $-a$ and a^{-1}, respectively.

THM C.2 Given $a, b \in F$, then the following statements are true:

- $a \cdot 0 = 0$
- $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$
- $(-a) \cdot (-b) = a \cdot b$

THM C.2. COR 1 The additive identity of a field has no multiplicative inverse.

DEF characteristic The characteristic of a field is defined as the smallest $p \in \mathbb{Z}^+$ s.t. $1 + 1 + \cdots + 1 = 0$ (p summands). Notes:

- If no integer exists, then field has characteristic zero.
- Fields of characteristic 2 are problematic.
- If $p \neq 0$, the sum $\forall x \in F$, $x + x + \cdots + x = 0$ (p summands).

2 1.2. Vector Spaces

DEF external binary NOT SURE ABOUT THIS DEF. IS IT $V \times F \to V$, or is more general than for v.s.es?

DEF vector space (v.s.) A vector space (linear space) V over a set F consists of a set on which two operations (addition (binary) and scalar multiplication (external binary)) are defined s.t. addition and scalar multiplication are closed over the v.s., and the following eight vector space axioms hold.

1. Commutativity of vector addition
2. Associativity of vector addition
3. Existence of the identity 0 for vector addition
4. Existence of the inverse for vector addition
5. $\forall x \in V,\ 1x = x$
6. $\forall a, b \in F, \forall x \in V,\ (ab)x = a(bx)$
7. $\forall a \in F, \forall x, y \in V,\ a(x+y) = ax + ay$
8. $\forall a, b \in F, \forall x \in V,\ (a+b)x = ax + bx$

THM 1.1. Cancellation Law for Vector Addition Let $x, y, z \in F$. Then $x + y = y + z \Rightarrow x = z$.

THM 1.1. COR 1 Uniqueness of the 0 vector.

THM 1.1. COR 2 Uniqueness of the additive inverse.

THM 1.2. Let V be a v.s. Then:

- $0x = 0$
- $(-a)x = -(ax) = a(-x)$
- $a0 = 0$

2.1 Examples of common v.s.

F^n set of n-tuples over elementwise addition and scalar multiplication

$M_{n \times m(F)}$ matrices over elementwise addition and scalar multiplication

$\mathbb{F}(S, F)$ set of functions mapping from a set S to a field F. Examples of this include sequences, which are $\mathbb{F}(\mathbb{Z}^+, F)$

$P(F)$, $P_n(F)$ set of polynomials of infinite or finite degree

3 1.3. Subspaces

DEF subspace A subset W of a v.s. V over a field F is a subspace of V if W is a v.s. over F with the $+, \cdot$ defined on V.

THM 1.3. Sufficient conditions for a subspace Let V be a v.s., $W \subseteq V$. Then W subsp. V IFF:

- $0 \in W$
- $x + y \in W\ \forall x, y \in W$
- $cx \in W\ \forall c \in F, \forall x \in W$

THM 1.4. Intersection of subspaces Any intersection of subspaces of a v.s. is a subspace.

3.1 Notes on subspaces

- For any v.s. V, $\{0\}$, V subsp. V.

3.2 Examples of subspaces

- The set of symmetric $n \times n$ matrices, diagonal $n \times n$ matrices, and matrices M s.t. trace$(M) = 0$ over F are subspaces of $M_{n\times n}(F)$.
- Even and odd functions in the set of functions over a field.
- P_n is the direct sum of the even and odd polynomials in P_n.

3.3 Results from the homework

20. Union of subspaces $W_1 \cup W_2$ subsp. V IFF $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

DEF set sum Let S_1, S_2 be non-empty subsets in V. The sum $S_1 + S_2 = \{x + y : x \in S_1, y \in S_2\}$.

DEF direct sum A v.s. V is the direct sum $V = W_1 \bigoplus W_2$ if W_1, W_2 subsp. V s.t. $W_1 \cap W_2 = \{0\}$, $W_1 + W_2 = V$.

23. Set sum of subspaces Let W_1, W_2 subsp. V. Then $W_1 + W_2$ subsp. V, and W_1, $W_2 \subseteq W_1 + W_2$. Also, any subsp. of V containing W_1 and W_2 contains $W_1 + W_2$.

30. Uniqueness of representation in direect sum If $W_1 \bigotimes W_2 = V$, then $\forall v \in V$, v can be uniquely be expressed as $w_1 + w_2,\ w_1 \in W_1,\ w_2 \in W_2$.

1. Null set vs. zero subspace The null set is never a subspace (all spaces must include the zero vector, so it is the smallest subspace)

9, 10. Good ways to check for subspaces:

- Check if the subspace has the zero vector. This is very easy to see and happens any time a linear combinations equals a nonzero vector.
- If the condition is that a linear combination equals zero, then it is the null space of a linear transformation, which is a v.s.

4 1.4. Linear Combinations and Systems of Linear Equations

DEF linear combination (l.c.) Let V be a v.s., $S \subseteq V$ nonempty subset of V. A vector $v \in V$ is a lin. comb. of vectors in S if there exist a finite number of vectors in S and a finite number of scalars in F s.t. $v = \sum_{i=1}^{n} a_n u_n$.

DEF span Let S be a nonempty subset of v.s. V. The span of S is the set of all lin. combs. of the vectors in S. Define span($\varnothing$) = $\{0\}$.

THM 1.5 The span of any subset S of a v.s. V subsp. V. Moreover, any subset of V containing S must also contain span(S).

DEF generate A subset S of a v.s. V generates (or spans) V if span(S)=V.

4.1 Notes about linear combinations

- 0 is a possible linear combination of any set.
- To find whether a vector is a linear combination of others, write it as lin. comb. and solve for coefficients w/ system of equations.

4.2 Results from the homework

17. Number of generating subsets $\text{card}(V) = n \iff V$ has finitely many generating subsets.

5 1.5. Linear dependence and linear independence

DEF linear dependence A subset S of a v.s. V is called linearly dependent if there exists a nontrivial lin. comb. over S that yields 0; i.e., there exists a nontrivial lin. comb. over S that equals 0.

DEF linear independence A subset S of a v.s. that is not linearly dependent is linearly dependent; i.e., lin. ind. IFF the only representations of 0 as linear combinations over S are trivial representations.

THM 1.6. Let V be v.s., $S_1 \subseteq S_2 \subseteq V$. If S_1 lin. dep, then S_2 lin. dep. Corollary (contrapositive): If S_2 lin. ind., then S_1 lin. ind.

THM 1.7. Let S be a lin. ind. subset of a v.s. V, let v be a vector in V not in S. Then $S \cup \{v\}$ lin. ind. IFF $v \in \text{span}(S)$.

5.1 Notes and results from the homework

- Any subset containing the 0 vector is linearly dependent.
- The empty set is linearly independent.
- A set consisting of a single nonzero vector is linearly independent.
- To show lin. ind., set lin. comb. of all vectors to 0, show all coefficients 0.

1a. Not every vector from a linearly dependent set may be expressable as a linear combination of the other vectors in that set.

6 1.6. Bases and dimension

DEF basis A basis β for a v.s. V is a lin. ind. subset of V s.t. $\text{span}(\beta) = V$.

THM 1.8 Let V be a v.s., $\beta = u_i$, $1 \leq i \leq n$. Then β basis for V IFF each $v \in V$ can be uniquely expressed as a lin. comb. of vectors of β.

THM 1.9 If a v.s. V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis. ("A finite spanning set can be reduced to a basis for V")

THM 1.10 Replacement theorem Let V be a v.s. generated by a set G containing n vectors, let L be a lin. ind. subset of V containing m vectors. Then $m \leq n$ and $\exists H \subseteq G$, $\text{card}(H) = n - m$ s.t. $\text{span}(L \cup H) = V$.

THM 1.10 Corollary 1 Dimension of bases Let V be a v.s. having a finite basis. Then every basis for V contains the same number of vectors.

DEF dimension A v.s. is finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V. A v.s. that is not finite-dimensional is called infinite-dimensional.

THM 1.10 Corollary 2 Let V be a v.s., $\dim(V) = n$. Then:

- Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.
- Any linearly independent subset of V that contains exactly n vectors is a basis for V.
- Every linearly independent subset of V can be extended to a basis for V.

(i.e., think: generating set always greater cardinality than lin. ind. set)

THM 1.11 Dimension of subspaces Let W subsp. finite-dimensional v.s. V. Then W is finite-dimensional, $\dim(W) \le \dim(V)$. If $\dim(W) = \dim(V)$, then $W = V$.

THM 1.11 Corollary 1 If W subsp. a finite-dimensional v.s. V, then any basis for W can be extended to a basis for V.

6.1 Example bases

- $\varnothing$ is the basis for $\{0\}$.
- $\{e_i\}$, $1 \le i \le n$ is the standard basis for F^n. Standard bases are also defined for $M_{m\times n}(F)$, $P_n(F)$.
- The Lagrange polynomials. STUDY THIS

6.2 Results from the HW

20. Extension of THM 1.9 to arbitrary subspaces Let V be a v.s. having dimension n, and let S be a subset of V that generates V. Then there is a subset of S that is a basis for V. (S can be infinite.)

24. Polynomial and derivatives If $f \in P_n(F)$ has degree n, then $\{f, f', f'', \cdots f^{(n)}\}$ is a basis for $P_n(F)$.

29. Dimension of set sum Two results:

- If W_1, W_2 finite-dimensional, subsp. V, then $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$.
- Let W_1, W_2 be finite-dimensional subspaces of v.s. V, let $V = W_1 + W_2$. Then $V = W_1 \bigoplus W_2 \iff \dim(V) = \dim(W_1) + \dim(W_2)$.

Examples 11, 12 from chapter The dimension of a v.s. may depend on its field; i.e., the v.s. of complex numbers may have dimension 1 over a field of $\mathbb{C}$, and dimension 2 over a field of $\mathbb{R}$.

6.3 Lagrange Interpolation Formula

Probably not on the tests, but a very interesting example nonetheless.

Given a finite set of distinct scalars $c_0, c_1, \cdots, c_n$ in an infinite field F. Then the polynomials $f_0, f_1, \cdots, f_n$, where $f_i(x) = \prod_{k=0,k\neq i}^{n} \frac{x-c_k}{c_i-c_k}$, are a basis for $P_n(F)$, and $f_i(c_j) = \begin{cases} 0 & i \neq j, \\ 1 & i = j \end{cases}$. Thus $\sum_{i=1}^{n} a_i f_j(c_j) = a_j$. We can show that $\{f_i\}$ is a basis by assuming $\sum_{i=1}^{n} a_i f_i = 0$; then, at each c_j we have an equation that asserts that c_j equals zero, i.e., $\sum_{i=1}^{n} a_i f_i(c_j) = 0 = c_j$. Since $\{f_i\}$ is a basis, then $\forall g \in P_n(F)$, $g(c_j) = \sum\limits_{i=1}^{n} b_i f_i(c_j) = b_j \Rightarrow g = \sum\limits_{i=1}^{n} g(c_i) f_i$. This can be applied to any set of ordered pairs of coordinates, $(c_i, g(c_i)),\ 1 \leq i \leq n$ to express (exactly) some polynomial g.

7 2.1. Linear Transformations, Null Spaces, and Ranges

DEF linear transformation Let V, W be v.s. over F. Then, the function $T : V \to W$ is called a linear transformation from V to W, if $\forall x, y \in F$, $\forall c \in F$, $T(x + y) = T(x) + T(y)$ and $cT(x) = T(cx)$).

Properties of a linear transformation Useful properties:

T linear, then $T(0) = 0$

- T linear IFF $T(cx + y) = cT(x) + T(y)$ (easier to check this than two things).
- The linear map over a linear combination preserves its structure, yielding a lin. comb. over the images of the vectors in the lin. comb with the same coefficients.

DEF kernel (null space) and range (image) Let V, W be v.s., $T : V \to W$ is linear. Define the null space (kernel) $N(T) = \{x \in V : T(x) = 0\}$. Also, define the range (image) $R(T) = \{T(x) : x \in V\}$.

THM 2.1 Let V, W be v.s., $T : V \to W$ linear. Then $R(T)$, $N(T)$ subsp. W, V, respectively.

THM 2.2 Let V, W be v.s., $T : V \to W$ linear, and β is a basis for V. Then $\text{span}(T(\beta)) = \text{span}(R(T))$.

DEF nullity, rank Let V, W be v.s., $T : V \to W$ linear. If $N(T)$, $R(T)$ finite dimensional, then define $\text{rank}(T) = \text{rank}(R(T))$, $\text{nullity}(T) = \text{rank}(N(T))$.

THM 2.3 Dimension Theorem Let V, W be v.s., $T : V \to W$ is linear. If V finite-dimensional, then $\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

THM 2.4 Null space of 1-1 transforms Let V, W be v.s. $T : V \to W$ linear. Then T is 1-1 IFF $N(T) = \{0\}$.

THM 2.5 Equivalence of 1-1, onto transforms Let V, W be v.s. of equal (finite) dimension, and let $T : V \to W$ be linear. Then the following are equivalent:

- T 1-1 (injective)
- T onto (surjective)
- $\text{rank}(T) = \dim(V)$

THM 2.6 Let V, W be v.s. over F, suppose $\{v_1, v_2, \cdots, v_n\}$ is a basis for V. $\forall w_1, w_2, \cdots, w_n \subseteq W$, $\exists!\ T \in \mathcal{L}(V, W) : T(v_i) = w_i$, $1 \le i \le n$.

THM 2.6 Corollary 1 Let V, W be v.s., suppose V has the basis $\{v_1, v_2, \cdots, v_n\}$. If $U, T \in \mathcal{L}(V, W)$ and $U(v_i) = T(v_i)$, $1 \le i \le n$, then $U = T$.

7.1 Examples of linear transformations

rotation in cartesian coordinates in $\mathbb{R}^2$ $T_\theta(a, b) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta)$

zero and identity transforms den. 0_V, 1_V or I_V

7.2 Notes from the homework

1. Finding a transform that maps a given set to a given image You cannot always find a linear map s.t. for some given $v_i \subset V$, $w_i \subset W$, $T(v_i) = w_i$.

2. 1-1 doesn't imply onto It is possible for two transforms to be onto but not 1-1; THM 2.5 only holds if the domain and codomain are of the same dimension.

DEF T-invariant Let V be a v.s, and let $T : V \to V$ be linear. A subspace W of V is said to be T-invariant if $T(x) \in W\ \forall x \in W$; i.e., $T(W) \subseteq W$. If W is T-invariant. we define the restriction of T on W to be the function $T_W : W \to W$, $T_W(x) = T(x)\ \forall x \in W$.

7.3 Notes on linear transforms

- (THM 2.5) doesn't apply for infinite-dimensional v.s.; i.e., onto isn't equivalent to 1-1 in this case. Also, (THM 2.4) and (THM 2.5) rely on linearity of T.
- (THM 2.6 Corollary 1) indicates that the mapping of any linear transform on the basis of its domain is unique. (THM 2.6) allows us to find a transform if a mapping of its basis is known, since it must be unique.

8 2.2 The Matrix Representation of a Linear Transformation

DEF ordered basis Let V be a finite-dimensional v.s. An ordered basis for V is a basis endowed with a specific order. For some v.ses, a standard ordered basis is defined. For our purposes, the choice of standard basis is arbitrary but must be consistent.

DEF coordinate vector Let $\beta = u_1, u_2, \cdots, u_n$ be an O.B. for a finite-dimensional v.s. V. $\forall x \in V$ x can be represented as a lin. comb. over β, i.e., $x = \sum_{i=1}^{n} a_i u_i$, $1 \leq i \leq n$. We define the coordinate vector of x relative to β as $[x]_\beta =$ HOW TO USE MATRICES

DEF matrix representation of a linear transform Let $\beta = v_1, v_2, \cdots, v_n$ be an O.B. for V, $\gamma = w_1, w_2, \cdots .w_n$ be an O.B. for W. Let $T \in \mathcal{L}(V, W)$. Then $T(v_j) = \sum_{i=1}^{n} a_{ij} w_i$. The matrix representation of T in the ordered bases β and γ is defined by $A_{ij} = a_{ij}$, and denote $A = [T]_\beta^\gamma$. In other words, it is the partitioned matrix $A = [\ [T(v_1)]_\gamma \mid [T(v_2)]_\gamma \mid \cdots \mid [T(v_n)]_\gamma\ ]$.

DEF addition and scalar multiplication over lin. trans. Define addition and multiplication over linear transforms as the addition and scalar multiplication in the field of the results of the transforms.

THM 2.7 Set of linear transf. is a vector space Let V, W be v.s. over F. Let $T : V \to W$ be linear. Then:

- $\forall a \in F,\ aT + U$ is linear.
- The collection of all linear transformations from V to W is a v.s. over F. We'll denote this $\mathcal{L}(V, W)$.

THM 2.8 Correspondence of lin. trans. with matrix representations Let V, W be finite-dim. v.s. with O.B.s β, γ, respectively, and let $T, U \in \mathcal{L}(V, W)$. Then:

- $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
- $[aT]_\beta^\gamma = a[T]_\beta^\gamma$

DO EXERCISES

9 2.3. Composition of Linear Transformations and Matrix Multiplication

THM 2.9 Linearity of composition of lin. trans. Let V, W, Z be v.s. over F, and let $T \in \mathcal{L}(V, W)$, $U \in \mathcal{L}(W, Z)$. Then $UT \in \mathcal{L}(V, Z)$.

THM 2.10 Properties of compositions of lin. trans. Let V be a v.s. Let $T, U_1, U_2 \in \mathcal{V}$. Then:

- $T(U_1 + U_2) = T(U_1) + T(U_2)$, $(U_1 + U_2)T = U_1 T + U_2 T$
- $T(U_1 U_2) = (TU_1)U_2$
- $TI = IT = T$
- $a(U_1 U_2) = (aU_1)U_2 = U_1(aU_2),\ \forall a \in F$

DEF matrix product Let $A \in M_{m\times p}(F), B \in M_{p\times n}(F)$. Define the product $AB \in M_{m\times n}(F)$ s.t. $(AB)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$.

THM 2.11 Matrix representation of composition Let V, W, Z be finite-dimensional v.s. with O.B.s α, β, γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations. Then $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.

Kronecker delta and I_n Define:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Define $I_n \in M_{n\times n}(F)$ s.t. $I_{ij} = \delta_{ij}$, and call this the n-by-n identity matrix.

THM 2.12 Properties of matrix multiplication Let $M \in M_{m\times n}(F)$, $B, C \in M_{n\times p}(F)$, $D, E \in M_{q\times m}(F)$. Then:

- $A(B + C) = AB + AC$, and $(D + E)A = DA + EA$
- $a(AB) = (aA)B = A(aB)\ \forall a \in F$
- $I_m A = A = AI_n$
- If V v.s. with dim. n and O.B. β, then $[I_V]_\beta = I_n$

THM 2.13 Let $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$. Let u_j be the jth column of AB, v_j be the jth column of B $\forall 1 \leq j \leq p$. Then:

- $u_j = Av_j$
- $v_j = Be_j$

THM 2.14 Linear transformations as matrix multiplication Let V, W be finite-dimensional v.ses with bases O.B.s β, γ, respectively. Let $T \in \mathcal{L}(V, W)$. Then, $\forall u \in V,\ [T(u)]_\beta^\gamma = [T]_\beta^\gamma [u]_\beta$.

DEF Left-multiplication Transformation Let $A \in M_{m\times n}(F)$. Define the transformation L_A, called the left-multiplication transformation, by $L_A : F^n \to F^m$, $L_A(x) = Ax\ \forall x \in F^n$.

THM 2.15 Properties of the left-multiplication transformation Let $A \in M_{m\times n}(F)$. Then L_A is linear. Furthermore, let $B \in M_{m\times n}(F)$, β, γ S.O.B.s for F^n, F^m, respectively, then:

- $[L_A]_\beta^\gamma = A$
- $L_A = L_B \iff A = B$
- $L_{A+B} = L_A + L_B$, and $L_{aA} = aL_A\ \forall a \in F$
- If $T : F^n \to F^m$ is linear, then $\exists!\, C$ s.t. $T = L_C$, and $C = [T]_\beta^\gamma$.
- If $E \in M_{n\times p}(F)$, then $L_{AE} = L_A L_E$.
- $m = n \Rightarrow L_{I_n} = I_{F^n}$.

THM 2.16 Let A, B, C be matrices s.t. $A(BC)$ is defined. Then so is $(AB)C$, and multiplication is associative.

9.1 Notes about matrix multiplication

- The transpose of a matrix product is the product of the transposes of the factors in reverse order, i.e., $(AB)^T = B^T A^T$.
- The cancellation law is not valid for matrices, since two different matrices may multiply the same matrix to get the same product. (E.g., some non-zero matrices square to 0.)

10 2.4. Invertibility and Isomorphisms

DEF invertible linear transformation Let V and W be v.ses, and let $T \in \mathcal{L})(V, W)$. A function $U \in \mathcal{L}(W, V)$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, it is said to be invertible, and its inverse is unique (by properties of invertible functions). Some properties for inverse linear transformations hold from the study of inverse functions (from Appendix B, can be used without proof):

- $(TU)^{-1} = U^{-1}T^{-1}$
- $(T^{-1})^{-1} = T$ (and thus T^{-1} is invertible)
- T is invertible IFF it is one-to-one (and thus onto, $\text{rank}(T) = \dim(V)$, etc.)

THM 2.17 Let V, W, be v.ses, and let $T \in \mathcal{L}(V, W)$ be invertible. Then T^{-1} is also linear.

DEF. invertible matrix Let $A \in M_{n\times n}(F)$. Then A is invertible if $\exists B \in M_{n\times n}(F)$ s.t. $AB = BA = I_n$.

LEM. Dimensions of domain and codomain of invertible transformations Let $T \in \mathcal{L}(V, W)$ invertible. Then V is finite-dimensional IFF W is finite-dimensional. In this case, $\dim(V) = \dim(W)$.

THM 2.18 Matrix representation of inverse transformation Let V, W be finite-dimensional v.ses with O.Bs β and γ, respectively. Let $T \in (V, W)$. Then T invertible IFF $[T]_\beta^\gamma$ invertible, and $[T]_\beta^\gamma = ([T^{-1}]_\gamma^\beta)^{-1}$.

THM 2.18 COR 2 Let $A \in M_{n\times n}(F)$. Then A invertible IFF L_A invertible, and $L_A^{-1} = L_{A^{-1}}$.

DEF isomorphic, isomorphism Let V, W be v.ses. We say that V is isomorphic to W if there $\exists T \in \mathcal{L}(V, W)$ invertible. T is called an isomorphism from V onto W.

THM 2.19 Let V, W be finite-dimensional v.ses. (over the same field). Then V isomorphic to W IFF $\dim(V) = \dim(W)$.

THM 2.20 Let V, W be finite-dimensional v.ses over F of dimensions n and m, respectively, and let β and γ be the respective O.Bs. Then the function $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$, defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism.

THM 2.19 COR 1 Let V, W be finite-dimensional v.ses of dimensions n, m, respectively. Then $\mathcal{L}(V, W)$ has dimension nm.

DEF standard representation function Let β be an O.B. for an n-dimensional v.s. V over the field F. The standard representation of V w.r.t. β is the function $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta \, \forall x \in V$.

THM 2.21 For any finite-dimensional v.s. V with O.B. β, ϕ_β is an isomorphism.

Commutativity diagram: performing matrices indirectly $\phi_\gamma T = L_{[T]_\beta^\gamma}\phi_\beta$. Note that ϕ_β, ϕ_γ invertible (but not necessarily T or L_A), so this has more possibilities open.

11 2.5. The Change of Coordinate Matrix

THM 2.22 Let β, β' be O.Bs for a finite dimensional v.s. V, and let $Q = [I_V]_{\beta'}^{\beta}$. Then:

- Q is invertible
- For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$

DEF linear operator A linear operator is a linear map with the same domain and codomain.

THM 2.23 Let T be a linear operator on a finite-dimensional v.s. V, and let β and β' be O.Bs for V. Suppose that Q is the change of coordinate matrix changing β' coordinates to β coordinates. Then $[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

THM 2.23 COR 1 Let $A \in M_{n\times n}(F)$, and γ be an O.B. for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.

DEF similar Let A, B be matrices in $M_{n\times n}(F)$. We say that B is similar to A if there exists an invertible matrix Q s.t. $B = Q^{-1}AQ$.

12 2.6. Dual Spaces (covered in brief)

DEF linear functional A linear functional is a linear transformation from a v.s. V to its field of scalars F.

DEF dual space For a v.s. V over F, define the dual space of V to be $\mathcal{L}(V, F)$, denoted by V^*.

13 3.1. Elementary Matrix Operations and Elementary Matrices

DEF elementary row [column] operation Let $A \in M_{m\times n}$. Any one of the following operations on the rows [columns] of A is called an elementary row [column] operation:

1. interchanging any two rows [columns] of A
2. multiplying any row [column] of A by a nonzero scalar
3. adding any scalar multiple of a row [column] of A to another row [column]

DEF elementary matrix An $n \times n$ elementary matrix is a matrix obtained by performing an elementary operation on I_n. The elementary matrix is called type 1, type 2, or type 3, depending on the type of elementary operation applied to it.

THM 3.1 Let $A \in M_{m\times n}(F)$, and suppose B is obtained from A by performing an elementary row [column] operation. Then there exists an $m\times m$ $[n\times n]$ elementary matrix E s.t. $B = EA$ $[B = AE]$. In fact, E is obtained from I_m $[I_n]$ by performing the same elementary row [column] operation performed on A to get B. The converse is also true.

THM 3.2 Elementary matrices are invertible, and the inverse of an elementary matrix is also an elementary matrix.

13.1 Notes about elementary matrices

- Each elementary matrix can be obtained by either a row or column operation.

14 3.2. The Rank of a Matrix and Matrix Inverses

DEF. rank If $A \in M_{m\times n}(F)$, we define $\text{rank}(A) = \text{rank}(L_A)$.

THM 3.3 Let $T \in \mathcal{L}(V, W)$ be a lin. trans. between finite-dim. v.ses, and let β, γ be respective O.Bs for V, W. Then $\text{rank}(T) = \text{rank}([T]_\beta^\gamma)$.

THM 3.4 Let $A \in M_{m\times n}$. If $P \in M_{m\times m}$, $Q \in M_{n\times n}$ invertible, then: $\text{rank}(PA) = \text{rank}(A) = \text{rank}(AQ) = \text{rank}(PAQ)$.

THM 3.4 COR 1 Elementary operations are rank-preserving.

THM 3.5 The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of $\text{Col}(A)$.

THM 3.6 Let $A \in M_{m\times n}(F)$ of rank r. Then $r \leq m, n$, and, by means of (a finite number of) elementary operations, A can be transformed into a matrix s.t. $A_{ij} = \begin{cases} 1 & i = j, i \leq r \\ 0 & \text{else} \end{cases}$

THM 3.6 COR 1 Let $A \in M_{m\times n}(F)$. Then $\exists B \in M_{m\times m}(F)$, $C \in M_{n\times n}(F)$ invertible s.t. BAC is the matrix described in the statement of the theorem.

THM 3.6 COR 2 For a matrix A, $\text{rank}(A) = \text{rank}(A^t) = \text{rank}(\text{Row}(A))$ ($= \text{rank}(\text{Col}(A))$).

THM 3.6. COR 3 Every invertible matrix is a product of elementary matrices.

THM 3.7 Let $T \in \mathcal{L}(V, W)$, $U \in \mathcal{L}(W, Z)$ on finite dim. v.s.es V, W, and Z, and let A, B be matrices s.t. AB is defined. Then:

- $\text{rank}(UT) \leq \text{rank}(U),\ \text{rank}(T)$
- $\text{rank}(AB) \leq \text{rank}(A),\ \text{rank}(B)$

DEF augmented matrix Let $A \in M_{m\times n}(F)$, $B \in M_{m\times p}(F)$. By the augmented matrix $(A|B)$, we mean the $m \times (n + p)$ matrix (AB).

Use of augmented matrix to find inverse If $A \in M_{m\times n}(F)$ invertible, and the augmented matrix $(A|B)$ is converted to $(I_n|B)$ by means of a finite number of elementary row operations, then $A^{-1} = B$. (If A is not invertible, then the rank will be less than n, which means that there will be a row of zeros on the left side).

15 3.3. Systems of Linear Equations – Theoretical Aspects

DEF system of m linear equations over n unknowns in the field F known

DEF coefficient matrix (of a system) known

DEF solution (of a system) known

DEF (in)consistent (describing a system) known

DEF homogeneous (describing a system) known

THM 3.8 Let $Ax = 0$ be a homogeneous lin. sys. of m lin. equations in n unknowns over a field F. Let K denote the set of all solutions to the system. Then $K = N(L_A)$; hence, K is a subspace of F^n of dimension $n - \text{rank}(L_A) = n - \text{rank}(A)$.

THM 3.8 COR 1 If $m < n$, then the system $Ax = 0$ has a nonzero solution.

THM 3.9 Let K be the sol'n set of a system $Ax = b$, and let K_H be a solution of the corresponding homogeneous system $Ax = 0$. Then for any solution s to $Ax = b$, $K = \{s\} + K_H = \{s + k : k \in K_H\}$.

THM 3.10 Let $Ax = b$ be a system of n linear equations in n unknowns. If A is invertible, then the system has exactly one solution, namely, $A^{-1}b$. Conversely, if the solution has exactly one solution, then A is invertible.

THM 3.11 Let $Ax = b$ be a system of linear equations. Then the system is consistent IFF $\text{rank}(A) = \text{rank}(A|b)$. (i.e., $b \in R(L_A)$).

16 3.4. Systems of Linear Equations – Computational Aspects

DEF equivalent systems Two systems of linear equations are called equivalent if they have the same solution set.

THM 3.13 Let $Ax = b$ be a system of m linear equations in n unknowns, and let C be an invertible $n \times n$ matrix. Then the system $(CA)x = Cb$ is equivalent to $Ax = b$.

THM 3.13 COR 1 Let $Ax = b$ be a system of m linear equations and n unknowns. If $(A'|b')$ is obtained from $(A|b)$ by a finite number of elementary row operations, then the system $A'x = b'$ is equivalent to the original system.

DEF reduced row echelon form A matrix is said to be in reduced echelon if the following three conditions are satisfied:

- Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).
- The first nonzero row in each row is the only nonzero entry in its column.
- The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

DEF Gaussian elimination back-substitution is more efficient than Gauss-Jordan elimination method

THM 3.14 Gaussian elimination transforms any matrix into its RREF form.

16.1 Notes about systems of linear equations

- Consistent solutions can have more than one, but a finite number of solutions (if a finite field).

17 Ways to prove v.s.

- Use 8 v.s. axioms.
- Show subspace of another v.s.
- Show it is R(T) or N(T) for some l.t. T.

18 Ways to prove basis

- Prove linearly independent, generating set.
- Prove linearly independent, correct dimension.
- Prove generating set, correct dimension.
- Image of a 1-1 linear transformation (i.e., isomorphism) on a basis.

19 Ways to show set equality

- Show set containment both ways

20 Methods for proofs

- Find all cases of (some condition): i.e., Be able to show IFF.
- Proof by contrapositive: $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$. Often useful for showing reverse direction in IFF.
- Induction: show base and inductive steps.

21 Questions

- Notation: $\{T(v)\}$ vs. $T(v)$ if v is a set of vectors?
- Notation: If $T \in \mathcal{L}(V, W)$, are $R(T)$ and $T(V)$ equivalent and/or interchangeable?
- When proving invertibility by finding an inverse and multiplying it, do you have to show that it multiplies to I_n with both left and right multiplication, or is one side sufficient?
- Is example in 2.5 backwards?