MA326 – CH4-6 Final Review

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4.1. Determinants of order 2

DEF. 2×2 **determinant** If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(F)$$

then we define the **determinant** of A, denoted $\det(A)$ or $|A|$ to be the scalar $ad - bc$.

THM 4.1. The function $det : M_{2\times 2}(F) \to F$ is a linear function of each row of a 2×2 matrix when the other row is held fixed. That is, if $u, v, w \in F^2$ and k is a scalar, then

$$\det \begin{pmatrix} u + kv \\ w \end{pmatrix} = \det \begin{pmatrix} u \\ w \end{pmatrix} + k\det \begin{pmatrix} v \\ w \end{pmatrix}$$

(and the same for the other row.)

THM 4.2. Let $A \in M_{2\times 2}(F)$. Then the determinant of $A \neq 0 \iff A$ invertible. If A invertible, then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Other results

Area of a parallelogram The determinant is equal to the area of a parallelogram (why?) formed by (u_1, u_2) and (v_1, v_2) is

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

This works for n-dimensional volumes of n-dimensional parallelpipeds.

4.2. Determinants of order n

DEF. higher-order determinant by cofactor expansion along the first row Let $A \in M_{n\times n}(F)$. If $n = 1$, then $|A| = (A_{11})$. See (THM. 4.4.) with $i = 1$. (This is not the most useful/general form.)

THM 4.3. This is an extension of (THM 4.1.) to higher-order matrices.

Corollary If a square matrix A has a row consisting entirely of zeroes, then $\det(A) = 0$.

THM 4.4. The determinant of a square matrix can be evaluated by cofactor expansion along any row i.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})$$

Corollary If $A \in M_{n\times n}(F)$ has two identical rows, then $\det(A) = 0$.

THM 4.5. Swapping two rows means flipping the sign of the determinant.

THM 4.6 Adding a multiple of one row to another doesn't change the determinant.

Corollary If $A \in M_{n\times n}(F)$ has rank less than n, then $\det(A) = 0$.

Other results

Determinant of an upper-triangular matrix The determinant of an upper-triangular matrix is the product of its diagonal entries.

4.3. Properties of determinants

THM 4.7. $\forall A, B \in M_{n\times n}(F)$, $\det(AB) = \det(A) \cdot \det(B)$.

Corollary A matrix $A \in M_{n\times n}(F)$ is invertible IFF $\det(A) \neq 0$. If A invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

THM. 4.8. $\forall A \in M_{n\times n}(F)$, $\det(A) = \det(A^T)$. This means that cofactor expansions can occur along any row or column (generalizing (THM 4.4.) even further).

THM 4.9. (Cramer's rule) Let $Ax = b$ be a linear system in n unknowns. If $\det(A) \neq 0$, then $x_k = \frac{\det(M_k)}{\det(A)}$., where M_k is the matrix obtained by replacing column k of A by b.

4.4. Summary – important facts about determinants

(Not much new is introduced in this section.)

- The determinant of similar matrices is the same.

5.1. Eigenvalues and Eigenvectors

DEF. diagonalizable A linear operator T on a finite-dimensional v.s. V is called **diagonalizable** if there is an O.B. β for V s.t. $[T]_\beta$ is a diagonal matrix. A square matrix A is called **diagonalizable** if L_A is diagonalizable.

DEF. e-val and e-vect Let T be a linear operator on a v.s. V. A nonzero vector $v \in V$ is called an **eigenvector** of T if $\exists \lambda \in F : T(v) = \lambda v$. The scalar λ is called the **eigevalue** corresponding to the eigenvector v.

THM 5.1. A linear operator T on a finite-dimensional v.s. V is diagonalizable IFF there exists an O.B. for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, v_2, \ldots, v_n\}$ is an O.B. of e-vects of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the e-val corresponding to v_j for $1 \leq j \leq n$,

THM 5.2. Let $A \in M_{n\times n}(F)$. Then a scalar λ is an e-val of A IFF $\det(A - \lambda I_n) = 0$.

DEF. characteristic polynomial Let $A \in M_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the **characteristic polynomial** of A. (Similar for T – characteristic polynomial is the characteristic polynomial of $[T]_\beta$).

THM 5.3. Let $A \in M_{n\times n}(F)$.

- The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.
- A has at most n distinct eigenvalues.

THM 5.4. Let T be a linear operator on a v.s. V, and let λ be an eigenvalue of T. A vector $v \in V$ is an e-vect of T corresponding to λ IFF $v \neq 0$ and $v \in N(T - \lambda I))$.

5.2. Diagonalizability

THM 5.5. Let T be a linear operator on a vector spacce V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct e-vals of T. If $v_1, v_2, \ldots, v_k$ are e-vects of T s.t. λ_i corresponds to $v_i (1 \leq i \leq k)$. then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Corollary Let T be a linear operator on an n-dimensional v.s. V. If T has n distinct e-vals, then T is diagonalizable.

THM 5.6. The characteristic polynomial of any diagonalizable linear operator splits.

DEF. Let λ be an e-val of a linear operator or matrix with characteristic polynomial $f(t)$. The **(algebraic) multiplicity** of λ is the largest positive integer k is the largest positive integer k for which $(t-\lambda)^k$ divides $f(t)$.

DEF. e-space Let T be a linear operator on a v.s. V, and let λ be an e-val of T. Define $E_\lambda = \{x \in T : T(x) = \lambda x\} = N(T - \lambda I_V)$. The set E_λ is called the **eigenspace** of T corresponding to the e-val λ.

THM. 5.7. Let T be a linear operator on a finite-dimensional v.s. V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \le \dim(E_\lambda) \le m$.

THM 5.8. Let T be a linear operator on a v.s. V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct e-vals of T. For each $i = 1, 2, \ldots, k$, let S_i be a finite linearly independent subset of the eigenspace E_{λ_i}. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of V.

THM. 5.9. Let T be a linear operator on a finite-dimensional v.s. V s.t. the characteristic polynomial T splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then

1. T is diagonalizable IFF the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for all i.
2. If T is diagonalizable and β_i is an O.B. for E_{λ_i} for each i, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an O.B. for V consisting of e-vects of T.

DEF. sum of subspaces Let $W_1, W_2, \ldots, W_k$ be subspaces of a v.s. V. We define the **sum** of these subspaces to be the set

$$\{v_1 + v_2 + \cdots + v_k : v_i \in W_i \ \forall 1 \le i \le k\}$$

which we denote by $\sum_{i=1}^k W_i$.

DEF. direct sum of subspaces Let $W_1, W_2, \ldots, W_k$ be subspaces of a v.s. V. We call V the **direct sum** of the subspaces of $W_1, W_2, \ldots, W_k$ and write $V = \bigoplus_{i=1}^k W_i$ if V is the sum of those subspaces and

$$W_j \cap \sum_{i \ne j} W_i = \{0\} \ \forall 1 \le j \le k$$

THM 5.10. Let $W_1, W_2, \ldots, W_k$ be subspaces of a finite-dimensional v.s. V. The following conditions be equivalent.

- V is the direct sum of subspaces $\{W_i\}$.

- $V = \sum_{i=1}^{k} W_i$, and for any vectors $v_1, v_2, \ldots, v_k$ s.t. $v_i \in W_i$, if $v_1 + v_2 + \cdots + v + k = 0$, then $v_i = 0 \ \forall i$.
- Each vector $v \in V$ can be uniquely written as $v = v_1 + v_2 + \cdots + v_k$, where $v_i \in W_i$.
- If γ_i is an O.B. for W_i, then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an O.B. for V.
- For each $i = 1, 2, \ldots, k$, there exists an O.B. γ_i for W_i s.t. $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an O.B. for V.

THM 5.11. A linear operator T on a finite-dimensional v.s. V is diagonalizable IFF V is the direct sum of the eigenspaces of T.

5.4. Invariant subspaces and the Cayley-Hamilton theorem

DEF. T-invariant subspace Let T be a linear operator on a v.s. V. A subspace W of V is called a $T-invariantsubspace$ of V if $T(W) \subseteq W$, that is, if $T(v) \in W \ \forall v \in W$.

DEF. T-cyclic subspace Let T be a linear operator on a v.s. V, and let x be a nonzero vector in V. The subspace $W = \text{span}(\{x, T(x), T^2(x), \ldots\})$ is called the **T-cyclic subspace** of V **generated by** x.

THM 5.21. Let T be a linear operator on a finite-dimensional v.s. V, and let W be a T-invariant subspace of V. Then the characteristic polynomial of T_W divides the characteristic polynomial of T.

THM 5.22. Let T be a linear operator on a finite-dimensional v.s. V, and let W denote the T-cyclic subspace of V generated by a nonzero vector $v \in V$. Let $k = \dim(V)$. Then

1. $\{v, T(v), \ldots, T^{k-1}(v)\}$ is a basis for W.
2. If $a_0 + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$, then the characteristic polynomial of T_W is $f(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$.

THM 5.23. Cayley-Hamilton. Let T be a linear operator on a finite-dimensional v.s. V, and let $f(t)$ be the characteristic polynomial of T. Then $f(T) = T_0$, the zero transformation. That is, T "satisfies" its characteristic equation.

Corollary Let $A \in M_{n \times n}(F)$, and let $f(t)$ be the characteristic polynomial of A. Then $f(A) = 0$.

THM 5.24. Let T be a linear operator on a finite-dimensional v.s. V, and suppose that $V = W_1 \bigoplus W_2 \bigoplus \cdots \bigoplus W_k$, where W_i is a T-invariant subspace of V for each i ($1 \leq i \leq k$). The characteristic polynomial of V is the product of the characteristic polynomials of the T-invariant subspaces.

DEF. direct sum of matrices Direct sum of matrices eww

THM 5.25. Let T be a linear operator on a finite-dimensional v.s. V. Let V be a direct sum. Then for $A = [T]_\beta$, $B_i = [T_{W_i}]_{\beta_i}$, then A is the direct sum of $\{B_i\}$.