Test 3 Outline
MA240 – Differential Equations

7.1. Definition of the Laplace Transform

- Definition 7.1.1.: Let f be a function defined for $t \geq 0$. Then $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ is the Laplace Transform of f.
- $\mathcal{L}$ is a linear transform
- Theorem 7.1.1. Transforms of basic functions
  - $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
  - $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
  - $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$
  - $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$
  - $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$
  - $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$
- Theorem 7.1.2.: Sufficient conditions for existence: If f is piecewise continuous on $[0, \infty)$ and of exponential order, then the Laplace transform of f exists for $s > c$
  - f of exponential order if $|f(t)| \leq Me^{ct}$, for $t > T$, M, c, T constants
- Theorem 7.1.3. $\lim_{s\to\infty} F(s) = 0$, assuming $F(s)$ exists

7.2. Inverse Transforms and Transforms of Derivatives

- Factor functions with distinct linear factors using partial fraction decomposition
- Theorem 7.2.2. Transform of a derivative: If f and first $(n-1)$ derivatives PC and of exponential order, then $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$

7.3., 7.4. Operational Properties of the Laplace Transform

- Theorem 7.3.1. Translation on s: $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
- Theorem 7.3.2.: Translation on t: $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$
  - Alternative form: $\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$ (only useful in forward direction)
  - To write a function using the unit step function, for each piecewise section $h(x)$ from a to b, add $h(t)(\mathcal{U}(t-a) - \mathcal{U}(t-b))$
  - $\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$
- Theorem 7.4.1. Derivatives of transforms: $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$
  - Because this is a multiplicative rule like shifts on s, can use either to do $\mathcal{L}\{t^n e^{at}\}$
  - Methodology for inverse:
    - Integrate $F(s)$ n times until you get a function $G(s)$ that you can take the inverse Laplace transform of
    - $\mathcal{L}^{-1}\{F(s)\} = (-t)^n g(t)$
  - Inverse useful when powers of almost-usable form in the denominator (e.g., $\frac{s}{(s^2+16)^2}$)
- Theorem 7.4.2. Laplace of convolution: $\mathcal{L}\{f \circledast g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\} = F(s)G(s)$

- $f \circledast g = \int_0^t f(\tau)g(t-\tau)d\tau$
    - If $g(t)=1$, then $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$ (Laplace of integral)
        - Useful in reverse form, can solve for integral when Laplace transform has a $\frac{1}{s}$ factor; then integrate $f(t)$
    - Useful for Volterra integral equations or interrodifferential equations
- Theorem 7.4.3. Transform of a periodic function: $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$

7.5. The Dirac Delta Function

- Unit impulse: $\delta_a(t-t_0) = \begin{cases} 0, \ 0 \le t < t_0 - a \\ \frac{1}{2a}, \ t_0 - a \le t < t_0 + a \\ 0, \ t \ge t_0 + a \end{cases}$
- Dirac delta function: $\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$
- Theorem 7.5.1. $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}, \ t_0 > 0$

7.6. Systems of Linear Differential Equations

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11.1. Orthogonal Functions

- Properties of the inner product (functional, complex analogue of dot product)
    - $(u,v)=(v,u)$ (commutativity)
    - $(ku,v)=k(u,v), \ k$ is a scalar (constants can be pulled out)
    - $(u,u)=0$ if $u=0$, $(u,u)>0$ otherwise (positivity)
    - $(u+v,w)=(u,w)+(v,w)$ (distributivity of dot product over addition)
    - $||\phi(x))||^2 = \int_a^b \phi^2(x)dx$
        - Can normalize a function by dividing by its norm
- Definition of inner product of functions: $(f_1,f_2) = \int_a^b f_1(x)f_2(x)\,dx$
    - Definition of orthogonality of functions: $(f_1,f_2)=0$
    - Note that the zero function is orthogonal to every function
    - A set of real-valued functions is an orthogonal set if every pair of functions in that set is orthogonal
        - An orthonormal set is an orthogonal set where $||\phi_n(x)||=1$
- Expressing vectors/functions in terms of orthogonal basis
    - Vector analogue: can use an orthogonal set of n vectors as a basis with which to express any n-space vector as a linear combination of them
        - To find coefficient of a basis vector, dot the entire expression with the basis vector and solve for the coefficient (which is also the projection): $c_n = \frac{\vec{u}\cdot\vec{v_n}}{||\vec{v_n}||^2}$
    - $f(x) = \sum_{n=0}^{\infty} \frac{(f,\phi_n)}{||\phi_n(x)||^2}\phi_n(x)$
        - This is called the orthogonal series expansion or the generalized Fourier series

- Definition of orthogonality with a weight function: ("orthogonal with respect to weight function $w(x)$) if $\int_a^b w(x)\phi_m(x)\phi_n(x)\,dx = 0,\ m \neq n$
  - In general, can include a weight function in an inner product – for our purposes, usually $w(x) = 1$

11.2. Fourier Series

- Definition: the Fourier series of a function f on the interval $(-p, p)$ is given by:
  - $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \sin\dfrac{n\pi x}{p} + b_n \sin\dfrac{n\pi x}{p}\right)$
  - $a_0 = \dfrac{1}{p}\int_{-p}^{p} f(x)\,dx$
  - $a_n = \dfrac{1}{p}\int_{-p}^{p} f(x)\cos\dfrac{n\pi x}{p}\,dx$
  - $b_n = \dfrac{1}{p}\int_{-p}^{p} f(x)\sin\dfrac{n\pi x}{p}\,dx$
- Definition: Piecewise continuous (PC) over a closed interval:
  - Has a finite number of jump discontinuities
  - f is continuous over each interval
- Convergence theorem for Fourier series: Let f and f' be PC on $[-p, p]$. For all x in $(-p, p)$, series converges at a point continuity. At a point of discontinuity the series converges to the average of the left- and right-hand limits
- $2p$ is the fundamental period of the sum; Fourier transform not only reflects function on $(-p, p)$ but also the periodic extension of f outside the interval.
  - At $x = p + 2n,\ n \in \mathbb{Z}$, converges to $\dfrac{f(+p-) + f(-p+)}{2}$ (average of left-hand limit of $x = p$ and right-hand limit of $x = -p$)

11.3. Fourier Cosine and Sine Series

- Even function can be represented with only a_0 term and a_n (cosine) terms
- Odd function can be represented with only b_n (sine terms)
  - Will converge to 0 at $x = -p, 0, p$
- Gibbs phenomenon (not covered in our class): overshooting of curve at a discontinuity; overshooting stays almost constant (doesn't go away) when $n \to \infty$, but width gets narrower
- Half-range extensions: for a function defined only over $(0, L)$:
  - 1. Can reflect the graph about the y-axis, now even
    - Choose $p = L$, now $b_n = 0$, $a_0 = \dfrac{2}{p}\int_0^L f(x)\,dx$, $a_n = \dfrac{2}{p}\int_0^p f(x)\cos\left(\dfrac{n\pi x}{p}\right)dx$; period is $2p$
  - 2. Can rotate the graph about the origin, now odd
    - Choose $p = L$, now $a_0 = a_n = 0$, $b_n = \dfrac{2}{p}\int_0^p f(x)\sin\left(\dfrac{n\pi x}{p}\right)dx$, period is $2p$
  - 3. Repeat function by defining $f(x + L) = f(x)$ on $(-L, 0)$

        - Choose $p = \frac{L}{2}$ and also integrate over $(0, L)$; $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right)\,dx$, same with b_n (procedure works out to be the same as doing even and odd half-range extensions), period is L
- Fourier series can be used as a solution to a DE where solution is periodic
    - Can use half-range extensions if only positive/negative domain known
    - Assume solution in the form $\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{p}\right)$, match coefficients

12.1. Separable Partial Differential Equations

- A partial differential equation (pde) given by the a function $u(x, y)$ and can any first or second partial derivatives of u
- Focus is on finding particular solutions to pdes (more useful in real-life applications)
- Method of separation of variables:
    - Write: $u(x, y) = X(x)Y(y)$ and substitute into original pde
    - Separate variables: now have some expression like $\frac{X'}{X} = \frac{Y'}{Y} = -\lambda$
        - Equal to some constant (λ is the separation coefficient) because the ratios are functions of two different variables; for them to be equal must be equal to (the same) constants
    - Rewrite as linear equations, and solve. The λ will lead to an eigenvalue problem. Solve for eigenfunctions of one variable using BVPs, and plug those into the second equation.
- General solution is the sum of all nontrivial component solutions (superposition principle)
- Classifying pdes:
    - hyperbolic if $B^2 - 4AC > 0$
    - parabolic if $B^2 - 4AC = 0$
    - elliptic if $B^2 - 4AC < 0$

12.2., 12.3., 12.4., 12.5. Classical PDEs

- Heat equation: $ku_{xx} = u_t,\ k > 0$
- One-dimensional wave equation: $a^2 u_{xx} = u_{tt}$
- Two-dimensional Laplace's equation: $u_{xx} + u_{yy} = 0$
- Boundary conditions (can specify any of these at a boundary):
    - Dirichlet condition: u
    - Neumann condition: u_x
    - Dirichlet condition: $u_x + hu$