Differential Equations – MA240
Test 2 Outline

4.1. Preliminary Theory

- Theorem 4.1.1. Existence of solutions for IVPs: If the coefficient functions $a_n(x) \cdots a_0(x)$ and $g(x)$ are continuous throughout the interval and $a_n(x) \neq 0$ throughout the interval, then a solution $y(x)$ exists on the interval and is unique.
- Boundary value problems have no existence theorem.
- To solve nth-order ODE, need to solve associated homogeneous equation first
- Differential operator (including differential polynomial operator) is linear
  - Can write DEs as $L(y) = 0$ or $L(y) = g(x)$
- Theorem 4.1.2. Superposition principle – Homogeneous Equations: linear combination of homogeneous solutions is also a solution to the homogeneous ODE
  - Constant multiple of solution to homogeneous ODE also solution.
  - $y = 0$ is always a solution to a homogeneous ODE.
- Definition 4.1.1. Linear dependence/independence: A set of functions is linearly independent if there exists a set of constants $c_1 \cdots c_n$, not all 0, such that a linear combination of the functions with the constants, then linearly independent; otherwise, linearly dependent
- Definition 4.1.2. Wronskian: If each of the functions $f_1(x) \cdots f_n(x)$, determinant of functions and their derivatives (up to $n-1$th derivative) is called the Wronskian
- Theorem 4.1.3. Set of solutions is linearly independent on I IFF $W \neq 0$ for every x in the interval
- Definition 4.1.3. Any set of n linearly independent solutions of the homogeneous nth-order linear ODE on I is called a fundamental set of solutions
- Theorem 4.1.4. There exists a fundamental set of solutions for the homogeneous nth-order linear ODE on the interval I
- Theorem 4.1.5. The general solution of the linear homogeneous ODE is $y = c_1 y_1(x) + \cdots + c_n y_n(x)$
- Any solution of a linear ODE free of arbitrary parameters is called a particular solution
- General solution of a linear ODE is $y = y_c + y_p$
- Theorem 4.1.7. Superposition principle for nonhomogeneous linear ODEs: If $L(y_{p_i}) = g_i(x)$, and $y_p = y_{p_1} + \cdots + y_{p_n}$, then $L(y) = g_1(x) + \cdots + g_n(x)$

4.2. Reduction of Order

- If one solution to linear homogeneous ODE known, then second solution can be found by substituting $y_2(x) = u(x) y_1(x)$.
  - $y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$
  - ***(know how to derive this one)***

4.3. Homogeneous Linear Equations with Constant Coefficients

- Auxiliary equation (in m)
- Three cases:
  - For distinct roots: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
  - For repeated roots: $y = c_1 e^{mx} + c_2 x e^{mx}$
  - For complex roots: $y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$

4.4. Undetermined Coefficients – Superposition Approach

- Approach to find y_p of a constant-coefficient non-homogeneous linear ODE with $g(x)$ being of constant, polynomial exponential, sin/cos, or product or linear combination of these forms
- Make sure to check the y_c to make sure there are no repeated terms; if there are, multiply by x
- For cosine and sine, make sure to have both cosine and sine terms in the y_p (because they generate one another as derivatives.)

4.6. Variation of Parameters

- No restrictions on finding y_p from the problem
- Based around finding $y_p(x) = u_1(x)y_1(x)$
- $u_i = \int \frac{W_i}{W} dx$
    - $y = -y_1 \int \frac{y_2(x)f(x)}{W} dx + y_2 \int \frac{y_1(x)f(x)}{W} dx$ for second-order
- Make sure ODE is in standard form before getting $f(x)$

4.7. Cauchy-Euler Equation

- $a_n x^n D^n y + a_{n-1} x^{n-1} D^{n-1} y + \cdots + a_0 y = 0$
    - Auxiliary equation $m(m-1)\cdots(m-n+1) + \cdots + 1 = 0$
- Three cases:
    - Distinct real roots: $y = c_1 x^{m_1 x} + c_2 x^{m_2 x}$
    - Repeated real root: $y = c_1 x^{mx} + c_2 x^{mx} \ln x$
    - Complex roots: $y = c_1 x^{\alpha x} \cos(\beta \ln x) + c_2 x^{\alpha x} \cos(\beta \ln x)$
- Reduction to constant-coefficients (generally easier to solve):
    - Substitution: $x = e^t$, $t = \ln x$
    - $\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dx}, \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$
    - Only works for solutions with $t > 0$; for negative solutions use $t = -x$

5.1. Linear Models: IVPs

- Free undamped motion: $m\ddot{x} + \omega^2 x = 0, \omega = \sqrt{\frac{k}{x}}$
    - This will yield a sinusoidal solution (constant coefficients, m has complex roots)
    - $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$
    - Alternate form with amplitude: $x(t) = A \sin(\omega t + \phi)$, $A = \sqrt{c_1^2 + c_2^2}$, $\phi = \tan^{-1} \frac{c_1}{c_2}$
- Free damped motion: $m\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0$, $2\lambda = \frac{\beta}{m}$
    - $m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$
    - Three cases:
        - $\lambda^2 - \omega^2 > 0$: overdamped
        - $\lambda^2 - \omega^2 = 0$: critically damped
        - $\lambda^2 - \omega^2 < 0$: underdamped
    - $x(t) = Ae^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t + \phi)$
- Driven damped motion: $m\ddot{x} + 2\lambda\dot{x} + \omega^2 x = f(x)$
    - Complimentary function is <u>transient term</u>, particular solution is <u>steady-state solution</u>
    - If period of driven motion is same as period of object's motion, then resonance occurs
- Series circuit analogue to damping: $L\ddot{q} + R\dot{q} + \frac{1}{C}q = E(t)$
    - If $E(t) = 0$, electrical vibrations of the circuit are <u>free</u>

- If $R = 0$, then simple harmonic motion
- Discriminant is $R^2 - \frac{4L}{C}$
  - $R^2 - \frac{4L}{C} > 0$: overdamped
  - $R^2 - \frac{4L}{C} = 0$: critically damped
  - $R^2 - \frac{4L}{C} < 0$: underdamped
- Steady-state current analogous to steady-state solution of motion

5.2. Linear models: BVPs

- $EI = \frac{d^4y}{dx^4} = 0$ for beam deflection
  - For boundary points:
    - Embedded: $y = 0, \frac{dy}{dx} = 0$
    - Free: $\frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} = 0$
    - Simply supported (hinged): $y = 0, \frac{d^2y}{dx^2} = 0$
  - $y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4$
- Eigenvalues and eigenfunctions
  - Two-point BVP involving linear ODE with parameter λ: try to find values that lead to any non-trivial solutions
  - e.g., $\ddot{y} + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$
    - for $\lambda \geq 0$, only trivial solution; for $\lambda < 0$, nontrivial solutions $\lambda = \left(\frac{n\pi}{L}\right)^2$; λ values that produce nontrivial solutions are called eigenvalues, and corresponding functions dependent on these eigenvalues are eigenfunctions (constant not important)

5.3. Nonlinear models

- E.g., nonlinear pendulum, estimate $\sin\theta \approx \theta$ in $l\frac{d^2\theta}{dt^2} = -g\sin\theta$
- E.g., catenary, $\frac{dy}{dx} = \frac{W}{T_1} = \frac{\rho s}{T_1}$, substitute $u = \frac{dy}{dx}$ (result is a hyperbolic cosine)
- ***E.g., rocket motion ******
- ***E.g., variable mass,*** $F = \frac{d}{dt}(mv)$***, determine formula for*** m***, do some substitutions***

6.1. Review of Series

- Ratio test to determine interval (and radius) of convergence
- Identity Property: If an infinite power theorem is equal to 0, then every coefficient c_n is 0
- A function is analytic at a point if it can be represented by a power series with $R > 0$ centered at that point (basically if differentiable)

- Sample Maclaurin series on right –>

Maclaurin Series	Interval of Convergence	
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$(-\infty, \infty)$	
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$(-\infty, \infty)$	
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$	
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$[-1, 1]$	(2)
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$(-\infty, \infty)$	
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$	
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$	$(-1, 1]$	
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1, 1)$	

6.2. Solutions about ordinary points

- Definition 6.2.1. A point $x = x_0$ is an ordinary point of the DE (in standard form) if both $P(x)$ and $Q(x)$ are analytic at x_0. A non-ordinary point is singular.
- Theorem 6.2.1. If $x = x_0$ is an ordinary point of the DE, can find two linearly-independent power series solutions of the form $y = \sum_{n}^{\infty} c_n (x - x_0)^n$ that converges at least on $|x - x_0| < R$, R is the distance to the closest singular point
- Solving an ODE is "the method of undetermined series coefficients," with a recurrence relation and using the identity property; then collect and group terms at the end
- This can work with nonpolynomial coefficients as well, using series multiplication

6.3. Solutions about singular points

- A singular point $x = x_0$ is regular if $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. If not, irregular.
- Theorem 6.3.1. Frobenius' Theorem: If $x = x_0$ is a regular singular point of the DE, there exists at least one equation of the form $y = \sum_{n}^{\infty} c_n (x - x_0)^{n+r}$. The series will converge on at least $0 < x - x_0 < R$.
  - Need to find r before solving recurrence relation
  - No assurance of two linearly-independent solutions
- Indicial equation in r, roots are solutions for r
- Three cases:
  - If $r_1 \neq r_2$ and differ by non-integer, then two linearly-independent solutions of the regular form.
  - If $r_1 \neq r_2$ and differ by integer, then y_1 of regular form, $y_2 = c y_1(x) \ln x + (\text{regular form})$, c can be 0
  - If $r_1 = r_2$, $y_2 = y_1(x) \ln x + (\text{regular form})$ (analogous to that of Cauchy-Euler with repeated roots)

Random things from discussion questions

- $D^n x^{n-1} = 0$, $D^n x^n = n!$
- $\ddot{y} + k^2 y = 0 \Rightarrow y = c_1 \cos kx + c_2 \sin kx$
- $\ddot{y} - k^2 y = 0 \Rightarrow y = c_1 \cosh kx + c_2 \sinh kx$