LINE AND SURFACE INTEGRALS
MA113 TEST 2 EQUATION SHEET/OUTLINE

16.1. Line Integrals

- If a curve C is smooth for $a \leq t \leq b$, then line integral over C exists
- To evaluate a line integral given as a parametric function of x, y, z
  $\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))|\vec{v}(t)|dt$
- Applications for objects defined along a curve:
  - Mass: $\int_C \delta ds$
  - First moments and COM: $M_{yz} = \int_C x\delta ds$, $\bar{x} = \dfrac{M_{yz}}{M}$, same with other moments and COM
  - Moments of inertia: $M_x = \int_C (y^2+z^2)\delta ds$, same with other moments of inertia
  - For a line integral on a plane (flat), line integral may be interpreted as the area of the "wall" created along the curve with a height $f(t)$, where $f(t)$ is the integrand
- If piecewise smooth function curve made of finite smooth curves, line integral over entire curve is equal to the sum of the line integrals of the curves
- Value of the line integral may be path-dependent

16.2. Vector Fields and Line Integrals: Work, Circulation, and Flux

- A <u>vector field</u> is a function that assigns a vector to each point on its domain, e.g., $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$
  - Continuous if component functions continuous, differentiable if component functions differentiable
- <u>Gradient field</u> is field of gradient vectors, shows direction of greatest increase of f
  - i.e., $\nabla f = \dfrac{\partial f}{\partial x}\hat{i} + \dfrac{\partial f}{\partial y}\hat{j} + \dfrac{\partial f}{\partial z}\hat{k}$
- Line integral of a curve in a vector field has the integrand being the scalar tangential component of F along C, or $\vec{F}\cdot\vec{T} = \vec{F}\cdot\dfrac{d\vec{r}}{ds}$, so $\int_C \vec{F}\cdot\vec{T}ds = \int_C \vec{F}\cdot\dfrac{d\vec{r}}{ds}ds = \int_C \vec{F}\cdot d\vec{r}$
  - To evaluate a line integral of a $\vec{F}(x,y,z)$ along $\vec{r}(t)$, express $\vec{F}$ in terms of t (i.e., $\vec{F}(x,y,z) \to \vec{F}(\vec{r}(t))$ by substituting functional components of $\vec{r}$ into functional components of $\vec{F}$) find $\dfrac{d\vec{r}}{dt}$, and evaluate $\int_C \vec{F}(\vec{r}(t))\cdot\dfrac{d\vec{r}}{dt}dt$
- If vector field $\vec{F}$ only has single component, then can express line integral wrt one coordinate
  - Define line integral of a function wrt one coordinate: $\int_C M(x,y,z)dx \equiv \int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}$ only contains the x component function M.
  - $\int_C M(x,y,z)dx + \int_C N(x,y,z)dy + \int_C P(x,y,z)dz = \int_C Mdx + Ndy + Pdz$ (i.e., sum of component line integrals is the total line integral)
- Applications of line integrals:
  - $W = \int_C \vec{F}\cdot\vec{T}ds$ (work is a regular line integral, $\vec{F}$ is force vector field
  - Flow $= \int_C \vec{F}\cdot\vec{T}ds$, $\vec{F}$ is velocity vector field (usually of a fluid)

- If the curve is closed (starts and ends in the same place, called circulation
  - Flux $= \int_C \vec{F} \cdot \vec{n} ds$, C is a simple (non-overlapping) closed curve, $\vec{F}$ is some field (fluid's velocity field, electric field, magnetic field), $\vec{n}$ is outward-pointing normal vector
    - Be careful about the signs: if curve traveling counterclockwise, then $\vec{n} = \vec{T} \times \hat{k}$, else switch order of vectors
    - For a closed curve counter-clockwise in the x-y plane, flux is $\oint_C M dy - N dx$

Summary of ways to indicate a line integral:

TABLE 16.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$

$\mathbf{W} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_C \mathbf{F} \cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$	Parametric vector evaluation
$= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$	Parametric scalar evaluation
$= \int_C M\, dx + N\, dy + P\, dz$	Scalar differential form

16.3. Path Independence, Conservative Fields, and Potential Functions

- $\vec{F}$ is a vector field defined in open region D in space; if for any two points A and B in D the line integral $\int_A^B \vec{F} \cdot d\vec{r}$ has the same value for any path, then the integral is path independent in D and $\vec{F}$ is conservative on D
  - Can represent integral with limits $\int_A^B$ instead of $\int_C$ to indicate path-independence
- If $\vec{F}$ is a vector field defined on D and $\vec{F} = \nabla f$ for some scalar function f on D, then f is called a potential function for $\vec{F}$
  - $\vec{F}$ is conservative $\Longleftrightarrow$ it is the gradient field of a potential function (see Theorem 2 below)
  - e.g., a gravitational potential is a scalar function whose gradient field is a gravitational field, same for electric potential, so gravitational and electric fields are conservative
- Assumptions necessary for conservative fields
  - Curves must be piecewise smooth (finitely-many smooth pieces connected end-to-end)
  - D is simply connected (every loop can be contracted to single point in D without ever leaving D)

- D is connected (two points in D can be connected without leaving D)
  - Note that simple-connectedness and connectedness do not imply one another *(why is this true? https://math.stackexchange.com/questions/729551/can-a-disconnected-set-be-simply-connected)*
- Theorem 1: Fundamental Theorem of Line Integrals (line integral analogue of FTC): Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $r(t)$. Let f be a differentiable function with a continuous gradient vector $F = \nabla f$ on a domain D containing C. Then $\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$
  - Proof: $\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$, which is equal to integrand of normal curve; then use FTC on this to show that $\int_C \vec{F} \cdot d\vec{r} = \int_A^B \frac{df}{dt} dt = f(B) - f(A)$
- Theorem 2: Conservative Fields are Gradient Fields: Let $\vec{F}(x, y, z)$ be a vector field whose components are continuous throughout open connected region D in space. Then $\vec{F}$ is conservative $\Leftrightarrow$ $\vec{F}$ is a gradient field ∇f for a differentiable function f.
  - Proof: $\vec{F} = \nabla f \Rightarrow \vec{F}$ is conservative is easy to prove because of Theorem 1: the integral over a gradient field is only dependent on the endpoints (therefore conservative)
  - Proof: $\vec{F}$ is conservative $\Rightarrow \vec{F} = \nabla f$: Show that $\frac{\partial f}{\partial x} = M$, same with other components (see p.923)
- Theorem 3: Loop Property of Conservative Fields: Equivalency of the statements:
  - $\oint_C \vec{F} \cdot d\vec{r} = 0$ around every loop in D
  - $\vec{F}$ is conservative on D
  - (these in turn are equivalent to $\vec{F} = \nabla f$)
- Component Test for Conservative Fields: Let vector field $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ on simply connected domain whose component functions have continuous first partial derivatives. Then $\vec{F}$ conservative IFF $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
  - Proof that these equations work (but not why they imply conservative-ness): write $\vec{F} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$, then solve for $\frac{\partial N}{\partial z}$ and other partial derivatives
- Finding potential functions: Once it is known that a field is conservative, then $\frac{\partial f}{\partial x} = M$ (and same for other components); differentiate to get components of f (i.e., $f = \int M dx\hat{i} + \int N dy\hat{j} + \int P dz\hat{k}$)
- Exact differential forms: $Mdx + Ndy + Pdz$ is an expression in differential form. It is exact if it is the total differential of some scalar function f over domain D. A differential can be checked for exactness just like component test for conservative fields
  - If line integral over conservative field written in differential form $\int_C Mdx + Ndy + Pdz$, can compute using method above for conservative fields

16.4. Green's Theorem in the Plane

- The divergence (flux density) of a vector field $\vec{F} = M\hat{i} + N\hat{j}$ at (x, y) is $\text{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$
  - Derivation: p. 932
  - Physical interpretation: similar to "expansion at a point": if larger vectors out than in, then positive divergence; if not, negative divergence; basically flux in an infinitesimal area (hence "flux density")
- The circulation density (k-component of curl, $\text{curl}\vec{F} \cdot \hat{k}$) of $\vec{F}$ at point (x, y) is $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$
  - k-component of the more general circulation field
  - Denotes spin (positive circulation density means counterclockwise) at a point
- Examples of divergence and circular density of certain vector fields:
  - Uniform expansion/compression: $\vec{F} = cx\hat{i} + cy\hat{j}$ constant divergence, no circulation density
  - Uniform rotation: $\vec{F} = -cy\vec{i} + cx\vec{j}$ 0 divergence, constant circulation density
  - Shearing flow: $\vec{F} = y\vec{i}$ 0 divergence, constant circulation density
  - Whirlpool: $\vec{F} = \frac{-y}{x^2 + y^2} i + \frac{x}{x^2 + y^2}\hat{j}$ 0 divergence, 0 circulation density
- Green's Theorem:
  - Theorem 4: Green's Theorem (Flux-Divergence or Normal Form): Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F}$ be a vector field in the plane, with components having continuous first partial derivatives in open region containing R. Then outward flux of $\vec{F}$ across C is:
    $\oint_C \vec{F} \cdot \vec{n} ds = \oint_C M dy - N dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} dx dy$, or the double integral of the divergence of the field over the region enclosed by the curve.
    - Makes sense – integrate "flux density" over a region to get flux
    - To remember this integral, think right side as "normal" integral of partials, left side as switch sign and multiply by $dxdy$
  - Theorem 5: Green's Theorem (Circulation-Curl or Tangential Form): (Same conditions as first part). Then ccw circulation is:
    $\oint_C \vec{F} \cdot \vec{T} ds = \oint_C N dy + M dx = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dx dy$
    - Integrate "circulation density" over a region to get circulation

- To remember this integral, think of left side as "normal line integral", think of right side as switch sign and divide by $dxdy$

- Taking flux (Thm. 4) of $\vec{G_1} = N\hat{i} - M\hat{j}$ gives circulation, taking circulation (Thm. 5) of $\vec{G_2} = -N\hat{i} + M\hat{j}$ gives flux, so closely related; either can be used to solve some problems by interchanging M and N (see p. 938)
- Proof on p. 939
- Can be used on any plane with a simply connected region, and also for some non-simply connected regions if same orientation of curves (see p. 940)

- Reverse Green's Theorem to find area: $\text{Area}R = \frac{1}{2}\oint xdy - ydx$
  - Derivation: $\text{Area} = \iint dydx = \iint \frac{1}{2} + \frac{1}{2}dydx = \oint \frac{1}{2}xdy - \frac{1}{2}ydx$ (or $\frac{\partial M}{\partial x} = \frac{1}{2}$, $\frac{\partial N}{\partial y} = \frac{1}{2}$)

16.5. Surfaces and Area

- Parametrization of a surface: $\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$
- A parameterized surface $\vec{r}(u,v)$ is smooth if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ are never 0 in the interior of the domain.
- The area of a smooth surface is $A = \iint_R |\vec{r}_u \times \vec{r}_v| dudv = \iint_R d\sigma$
- For an implicit surface $F(x,y,z) = c$ over closed and bounded region, assume $\nabla F \neq 0$, $\nabla F \cdot \vec{p} \neq 0$ ($\vec{p}$ is unit vector normal to plane "shadow," so never folds back on itself), and smooth
  - $d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}dxdy$
  - surface area of an implicit function is $\iint_R d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}dA$, where $\vec{p}$ normal to R, $\nabla F \cdot \vec{p} \neq 0$
- $d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$, for surface defined by $z = f(x,y)$

16.6. Surface Integrals

- The surface integral of a scalar function $f(x,y,z)$ is $\iint_S f(x,y,z)\,d\sigma$, where $d\sigma$ is one of the differential forms from 16.5

- Orientation of a surface
    - A smooth surface S is <u>orientable</u> or <u>two-sided</u> if it is possible to define a field $\vec{n}$ of unit normal vectors on S that varies continuously with position
        - Smooth closed surfaces are orientable
        - $\vec{n}$, by convention, points outwards from a closed surface
        - The Mobius band is not orientable
- $\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$
    - Can be positive or negative depending on orientation (not very important)
    - For a surface given parametrically, flux is $\iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, du\, dv = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ du\, dv$
    - For a surface given implicitly, flux is: $\iint_S \vec{F} \cdot \frac{\nabla g}{|\nabla g|} \frac{|\nabla g|}{|\nabla g \cdot \vec{p}|} = \iint_S \vec{F} \cdot \frac{\nabla g}{|\nabla g \cdot \vec{p}|}\, dA$
- Applications of surface integrals
    - Same as 1D, 2D analogues, replace integral with $\iint_S$ and delta with $d\sigma$

16.7. Stokes' Theorem

- Let S be a piecewise smooth surface having a piecewise smooth boundary C. Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\vec{F}$ around C in the direction counterclockwise wrt unit normal vector $\hat{n}$ is $\nabla \times \vec{F} \cdot \hat{n}$ over S.
    - $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma$
- If two different oriented surfaces have the same boundary, they have the same curl integral
- For a two-dimensional field (Green's Theorem):
    - $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F} \cdot \hat{k}\, dA$ (i.e., the $\hat{k}$-component of curl)
        - $\nabla \times \vec{F} \cdot \hat{k}$ is the new circulation density, is equal to $\frac{1}{\text{Area}} \oint_C \vec{F} \cdot d\vec{r}$
- Circulation over a curve is the flux of the curl across a surface bounded by that curve, as long as the curves are traced in the same orientation (i.e., all curves have the surface to the left of them
- $\text{curl grad } f = \vec{0}$, or $\nabla \times \nabla f = \vec{0}$

- For a simply-connected open region D, $\nabla \times \vec{F} = \vec{0} \Rightarrow \oint_C \vec{F} \times d\vec{r} = 0 \Rightarrow$ field is conservative over D

16.8. Divergence Theorem

- Divergence Theorem: $\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV$
- For the field $\vec{F} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\rho^3}, \rho = \sqrt{x^2 + y^2 + z^2}$
  - For any region between two spherical shells, $\iiint_D \nabla \cdot \vec{F}\, dV = 0$
  - For any sphere, flux is 4π
- Gauss's Law: for any region encompassing the origin, $\iint_S \vec{E} \cdot \vec{n}\, d\sigma = \dfrac{q}{\epsilon_0}$ (p. 979)
- Continuity equation of hydrodynamics: $\nabla \cdot \vec{F} + \dfrac{\partial \delta}{\partial t} = 0$ (p. 979)
- Unifying Fundamental Theorem: the integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region