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MA223
Vector Calculus
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Chapter 15 Outline

Vector Calculus Test 1 Review

15.1: Multiple Integrals

- Partition a rectangular region R into small n rectangles in the x and y direction. Each subrectangle R_i has dimensions $(\Delta x, \Delta y)$ and area $\Delta A = \Delta x \Delta y$. Then, sum of values is $S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A$
- limit of S_n is the double integral: $\lim_{\|P\|\to 0} S_n = \iint_R f(x,y)dA = \iint_R f(x,y)dxdy$
- value of double integral can be interpreted as volume under surface $z = f(x,y)$
- Fubini's theorem states that double integrals can be evaluated as iterated integrals (in either order) for rectangle $R : x \in [a,b], y \in [c,d]$
  - $\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$

15.2: Double Integrals over General Regions

- Fubini's theorem for functional limits (in one dimension):
  - For $R : x \in [a,b], y \in [g_1(x), g_2(x)]$: $\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$
  - For $R : x \in [h_1(y), h_2(y)], y \in [c,d]$: $\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$
- Sometimes it's easier to evaluate double integrals when order of integration is reversed: to do this, draw the region and express the limits as a function of the other variable.
- Properties of double integrals: scalars can be taken out, sum/difference is distributive over integration, domination of one integral over another if the first's integrand is greater for every value in the domain than the second's integrand, and additivity of integrals into the union of their regions if their regions are mutually exclusive.

15.3: Area by Double Integration

- The area of a closed, bounded plane region R is $\iint_R dA$ (literally sum of the area differentials)
- Average value of f over R: $\frac{\text{total value}}{\text{total area}} = \frac{\iint_R f(x,y)dA}{\iint_R dA}$

15.4: Double Integrals in Polar Form

- Partition a polar region into sectors of angle $\Delta\theta$, and then partition sectors into subregions of length Δr. Each subregion can be approximated with an area of length $r\Delta\theta$ and width Δr. Thus, $S_n = \sum_{k=1}^{n} f(x_k, y_k) r\Delta\theta$.
- $\lim_{\|P\|\to 0} S_n = \iint_R f(r,\theta)dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r dr d\theta$ for $R: \theta \in [\theta_1, \theta_2], r \in [r_1(\theta), r_2(\theta)]$.
  By Fubini's theorem, if θ is bounded by functions of r (unlikely but possible), the limits of integration can be switched. (And if both r and θ are bounded by constants, the order of integration doesn't matter.)
- Remember rules/tricks of changing polar equations to Cartesian ones:
  - $r^2 = x^2 + y^2$
  - $x = r\cos\theta$, $y = r\sin\theta$
  - $\tan\theta = \frac{y}{x}$
- Polar equations can be used to solve equations that are given in Cartesian form but contain some of the above equations, e.g., $\iint_R e^{x^2+y^2} dA$.

15.5: Triple Integrals in Rectangular Coordinates

- Partition a solid into small rectangular prisms of dimensions Δx, Δy, and Δz. The volume of each subspace is $\Delta V = \Delta x \Delta y \Delta z$. For a function $f(x,y,z)$ defined over the space D, $S_n = \sum_{k=1}^{n} f(x,y,z)\Delta V$.
- Limit of S_n is the triple integral $\iiint_D f(x,y,z)dV = \iiint_D f(x,y,z)dxdydz$ and can be evaluated with an iterated integral.
- Volume of a space D: $\iiint_D dA$
- Using functional limits: $\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)dzdydx$
  - i.e., successive limits of integration can be functions that depend on the variables of outer integrals (same as double integral with Fubini's theorem)
- Average value of $f(x,y,z)$ over D: $\frac{\text{total value}}{\text{total volume}} = \frac{\iiint_D f(x,y,z)dV}{\iiint_D dV}$
- Triple integrals have same properties as double integrals.

15.6: Moments and Centers of Mass

- Mass and moments analogous to one- and two-dimensional counterparts:
  - Mass: $M = \iiint_D \sigma(x,y,z)dm$, $\sigma(x,y,z)$ is the density function
  - Moment along x-axis: $M_{yz} = \iiint_D x\sigma(x,y,z)dm$, analogous for moments along y (M_{xz}) and z (M_{xy}) axes

- Center of mass in x direction: $\bar{x} = \frac{M_{yz}}{M}$, analogous for COM in y and z directions
- Moment of inertia in one dimension: $I = \int r^2 dm$, $r = |x - \bar{x}|$
  - $KE_{shaft} = \frac{1}{2} I \omega^2$
  - For an object D, moment of inertia about L is $I_L = \iiint\limits_D r^2 dm = \iiint\limits_D r^2 \sigma(x, y, z) dV$, r is distance of a point $P(x, y, z)$from L
  - For two-dimensional plate, “polar moment” (moment about the origin) is $I_0 = \iint\limits_R (x^2 + y^2)\delta = I_x + I_y$

15.7: Triple Integrals in Cylindrical Coordinates

- A point in space can be represented in cylindrical coordinates $P(r, \theta, z)$:
  - r is length of projection of $\overrightarrow{OP}$ on the xy-plane
  - θ is the angle between the projection of $\overrightarrow{OP}$ on the xy-plane and the x-axis
  - z is the height of P (same as Cartesian coordinates)
- Constant parameter interpretations for function $f(r, \theta, z)$:
  - Constant r means f lies on a circular cylinder
  - Constant θ means f lies on a plane parallel to the z-axis
  - Constant z means f lies on plane parallel to the xy-plane
- For integral of space over cylindrical coordinates, partition first by angle θ, then by radius, then by height. Each subspace D_k is approximately a rectangular prism with dimensions $(r\Delta\theta, \Delta r, \Delta z)$ and volume $V_k = r\Delta r \Delta\theta \Delta z$. The integral is $\iiint\limits_D f(r, \theta, z) r dr d\theta dz$.

15.7: Triple Integrals in Spherical Coordinates

- A point in space can be represented in spherical coordinates $P(\rho, \theta, \phi)$:
  - ρ is $\left\|\overrightarrow{OP}\right\|$
  - θ is same as cylindrical coordinates
  - ϕ is angle between $\overrightarrow{OP}$ and the z-axis
- Useful equations for spherical coordinates:
  - $r = \rho \sin\phi$
  - $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$
  - $z = \rho \cos\phi$
  - $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$
- Constant parameter interpretations for function $f(\rho, \theta, \phi)$:
  - Constant ρ means f lies on a sphere
  - Constant θ means f lies on a plane
  - Constant ϕ means f lies on a cone
- For integral of space over spherical coordinates, partition first by angle θ, then by radius, then by ϕ . Each subspace D_k is approximately a rectangular prism with dimensions $(\rho\Delta\theta, \rho\Delta\phi, \Delta\rho)$ and volume $V_k = \rho^2 \Delta\rho \Delta\theta \Delta\phi$. The integral is $\iiint\limits_D f(\rho, \theta, \phi) \rho^2 d\rho d\theta d\phi$.