NAME ________________________

**Group Theory: An Introduction** (feat. the Rubik's Cube Group)

Vocabulary sheet and presentation by Jonathan Lam

| Term | Notation | Definition |
|---|---|---|
| group | (A, *) | an algebraic structure comprising a set and an operation that follow the group axioms |
| underlying set | A = { 1, a, b, c, … } | the set (list of elements) of a group |
| group law | * | a binary operation (an operation acting on two elements) |
| group axioms | | closure, associativity, identity, invertibility |
| closure | | applying the group rule on two elements of the set results in another element of the set |
| associativity | a + (b + c) = (a + b) + c<br>(or any other grouping) | changing the grouping of operations does not change the result |
| abelian group | a + b = b + a<br>for all a, b, c ∈ A | a group with a commutative group rule (changing the order of elements does not change the result) |
| identity element | 1<br>(not literally the value 1) | an element of a group's set that results in the other operand when the group rule is performed on it; this must exist in every group |
| inverse element | $a^{-1} * a = 1$<br>for all a ∈ A | when the group rule is applied on an element and its inverse, the identity element is obtained; all elements in a group must also have an inverse in the set |
| cardinality (order of a set) | \|A\| | the number of elements in a set |
| period (order of an element) | n such that $a^n = 1$ | the number of times the group rule must be applied on an element repeatedly until the identity element is obtained |
| subgroup | | a group with the same operation as its overgroup whose elements are all part of its overgroup's set |

More depth past the presentation's content for fun (because 10 minutes is waaaay too short):

More about groups:

- Groups are a limited part of Set Theory, dealing only with binary operations and obeying the group axioms; sets in general can have larger n-ary operations (e.g., ternary, quaternary, etc.) and do not have to obey the group axioms
  - That being said, groups are specifically designed to highlight properties of symmetry
- A cardinal number is a number defined as the the size of a set (a.k.a., order of a set), and can be either finite or transfinite (for infinite sets, see below)
  - Cardinality for finite sets with the same number of elements are equal natural numbers
  - Cardinality for infinite sets may not be the same, because they are transfinite: larger than any finite number, but smaller than absolute infinity
  - Cardinality for the subset of an infinite set can be equal to the cardinality of the original set
  - The cardinality of the Rubik's cube set is $\frac{12!\times 2^{11}\times 8!\times 3^{7}}{2} \approx 4.3\times 10^{19}$, due to the fact that there are 12 permutations for corners (12!), 2 orientations for corners except the last one ($2^{11}$), 8 permutations for edges (8!), 3 orientations for edges except the last one ($3^{7}$), and an even number of cube swaps ($2^{-1}$)
- The order of an element of a set (a.k.a., period of an element) (not to be confused with order of a set) is the number of times the operation will be repeated on the element until the identity element is attained; if it never reaches the identity element, its order is infinite
  - For example, the order of R in the Rubik's cube (G, ·) is four, because you can perform the R turn four times to get back to the solved state
  - For example, the order of 1 in (R, +) is infinite, because constantly adding 1 to itself will never equal zero
  - The largest order of any element in the Rubik's cube is 1260; there are multiple elements that achieve this, but one simple one is $RU^{2}D'BD'$
  - The order of the identity element is always 1 (intuitively)
- While the group rule can be represented in many ways, the most general representations include the multiplicative group and additive group representations (because addition and multiplication are both simple examples of groups)
  - Multiplicative: ab and a*b (mimicking multiplication), in which the identity element is represented as 1 and an exponent indicates how many times it is applied to itself; its order is usually defined as the n such that $a^{n} = 1$ (or infinity for an infinite order); the inverse is represented $a^{-1}$, and the exponent representation remains
  - Additive: a+b, in which the identity element is represented as 0 and the order is the n such that na = 0. The inverse is represented as -a.
- In symmetry groups, the arrangement of the set elements may seem reversed, but this is because of function composition notation

    - For example, (f · g) (a) means f(g(a)), thus g is the first operation performed, and f is second
    - Similarly, the ordinary, intuitive Rubik's cube notation would be represented backwards in correct function composition notation: RU becomes the element U·R
- In Graph Theory, a Hamiltonian path is an idea in graph theory that passes through each point exactly once, and can be used to generate an algorithm that cycles through every element of the Rubik's cube set
- The trivial group is a group with only one element in its set: the identity element
- An important part of groups is studying homomorphisms: groups with similar structures
    - Isomorphisms are bijective (one-to-one) homomorphisms
    - Automorphisms are homomorphisms over an own set

Further reading: academic reports specific to the study of Rubik's Cubes through Group Theory:

Chen, Janet. *Group Theory and the Rubik's Cube*. N.p., n.d. Web. 25 May 2018. <http://www.math.harvard.edu/~jjchen/docs/Group%20Theory%20and%20the%20Rubik's%20Cube.pdf>.

Daniels, Lindsey. *Group Theory and the Rubik's Cube*. Lakehead University, N.d. Web. 25 May 2018. <http://math.fon.rs/files/DanielsProject58.pdf>.

Davis, Tom. *Group Theory vis Rubik's Cube*. geometer.org, 6 Dec. 2006. Web. 25 May 2018. <http://www.geometer.org/rubik/group.pdf>.

Howell, Zeb. *Explorations of the Rubik's Cube Group*. N.p., 18 Apr. 2016. Web. 25 May 2018. <http://buzzard.ups.edu/courses/2016spring/projects/howell-rubiks-cube-ups-434-2016.pdf>.

Rubik's cubes:

- How to solve the Rubik's cube: https://ruwix.com/
- Get a decent speedcube for $2.99: https://thecubicle.us/yuxin-little-magic-p-9649.html